ROTATIONAL MOTION

So far we have learnt kinematics and dynamics of translation motion in which all the particles of a body undergo identical motions i.e. at any instant of time all of them have equal velocities and equal accelerations and in any interval of time they all follow identical trajectories. Therefore kinematics of any particle of a body or of its centre of mass in translation motion is representative of kinematics of the whole body. But when a body is in rotational motion, all its particles and the centre of mass do not undergo identical motions. Newton's laws of motion, which are the main guiding laws of mechanics, are applicable to a particle and if applied to a rigid body or system of particles, they predict the motion of the centre of mass. Therefore, it becomes necessary to investigate how the centre of mass and different particles of a rigid body move when a body rotates.

1. RIGID BODY

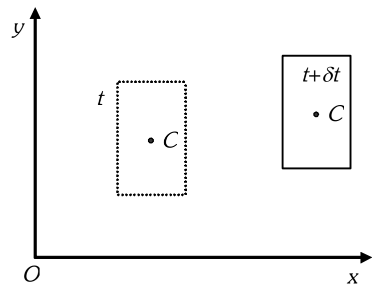
A rigid body is an assemblage of a large number of material particles, which do not change their mutual distances under any circumstance or in other words, the body is not deformed under any circumstance.

Actual material bodies are never perfectly rigid and are deformed under the action of external forces. When these deformations are small enough not to be considered during the course of motion, the body is assumed to be a rigid body. Hence, all solid objects such as stone, ball, vehicles etc are considered as rigid bodies while analyzing their translational as well as rotational motion.

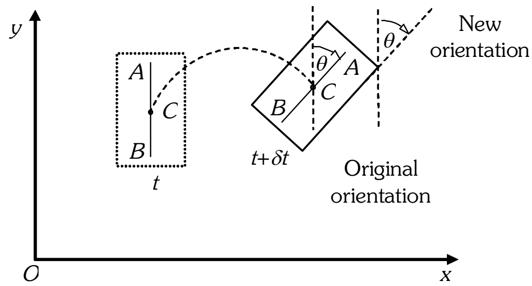
2. ROTATIONAL MOTION OF A RIGID BODY

Any kind of motion is identified by change in position or change in orientation or change in both. If a body changes its orientation during its motion it said to be in rotational motion.

In the following figures, a rectangular plate is shown moving in the x-y plane. The point C is its centre of mass. In the first case it does not change its orientation, therefore is in pure translation motion. In the second case it changes its orientation during its motion. It is a combination of translational and rotational motion.



Pure Translation



Combination of translation and rotation

Rotation i.e. change in orientation is identified by the angle through which a linear dimension or a straight line drawn on the body turns. In the figure this angle is shown by θ.

2.1 Types of Motions involving Rotation

Motion of body involving rotation can be classified into following three categories.

I Rotation about a fixed axis.

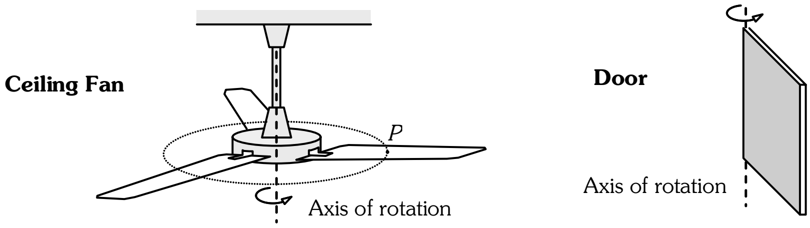
II Rotation about an axis in translation.

III Rotation about an axis in rotation

Rotation about a fixed axis

Rotation of ceiling fan, opening and closing of doors and rotation of needles of a wall clock etc. come into this category.

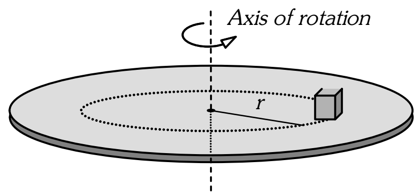
When a ceiling fan rotates, the vertical rod supporting it remains stationary and all the particles on the fan move on circular paths. Circular path of a particle P on one of its blades is shown by dotted circle. Centres of circular paths followed by every particle on the central line through the rod. This central line is known as the axis of rotation and is shown by a dashed line. All the particles on the axis of rotation are at rest, therefore the axis is stationary and the fan is in rotation about this fixed axis.



A door rotates about a vertical line that passes through its hinges. This vertical line is the axis of rotation. In the figure, the axis of rotation is shown by dashed line.

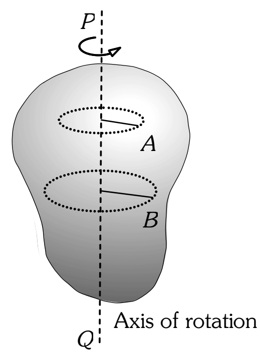
Axis of rotation

An imaginary line perpendicular to the plane of circular paths of particles of a rigid body in rotation and containing the centres of all these circular paths is known as axis of rotation.



It is not necessary that the axis of rotation should pass through the body. Consider a system shown in the figure, where a block is fixed on a rotating disc. The axis of rotation passes through the center of the disc but not through the block.

Important observations



Let us consider a rigid body of arbitrary shape rotating about a fixed axis PQ passing through the body. Two of its particles A and B are shown moving on their circular paths.

All its particles, not lying on the axis of rotation, move along circular paths with centres on the axis or rotation. All these circular paths are in parallel planes that are perpendicular to the axis of rotation.

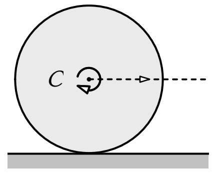
All the particles of the body undergo same angular displacement in the same time interval, therefore all of them move with the same angular velocity and angular acceleration.

Particles moving on circular paths of different radii move with different speeds and different magnitudes of linear acceleration. Furthermore, no two particles in the same plane perpendicular to the axis of rotation have same velocity and acceleration vectors.

Rotation about an axis in translation

Rotation about an axis in translation includes a broad category of motions. Rolling is an example of this kind of motion.

Consider the rolling of wheels of a vehicle, moving on straight levelled road.

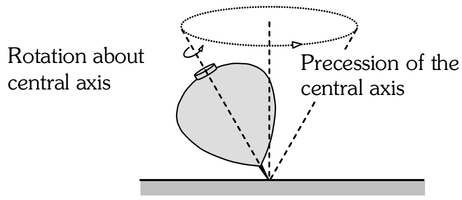


The wheel appears rotating about its stationary axle relative to a reference frame, attached with the vehicle. The rotation of the wheel as observed from this frame is rotation about a fixed axis. Relative to a reference frame fixed with the ground, the wheel appears rotating about the moving axle, therefore, rolling of a wheel is superposition of two simultaneous but distinct motions – rotation about the axle fixed with the vehicle and translation of the axle together with the vehicle.

Rotation about an axis in rotation.

In this kind of motion, the body rotates about an axis which in turn rotates about some other axis. Analysis of rotation about rotating axes is beyond our scope, therefore we shall keep our discussion elementary level only.

As an example consider a rotating top. The top rotates about its central axis of symmetry and this axis sweeps a cone about a vertical axis. The central axis continuously changes its orientation, therefore it is in rotational motion. This type of rotation in which the axis of rotation also rotates and sweeps out a cone is known as *precession*.



Another example of rotation about an axis in rotation is a swinging table-fan while running. Table-fan rotates about its shaft along which its axis of rotation passes. When running swings, its shaft rotates about a certain axis.

3. KINEMATICS OF ROTATIONAL MOTION

Angular Displacement (θ)

- When a particle moves in a curved path, the change in the angle traced by its position vector about a fixed point is known as angular displacement.
- Unit : radian
- Dimension : $M^0L^0T^0$ i.e. dimensionless.
- Elementary angular displacement is a vector whereas other angular displacements is a scalar.

Angular Velocity (ω)

- The angular displacement per unit time is defined as angular velocity.

$\omega = \dfrac{\Delta\theta}{\Delta t}$, where $\Delta\theta$ is the angular displacement during the time interval Δt.

- Instantaneous angular velocity $\omega = \lim\limits_{\Delta t \to 0}\dfrac{\Delta\theta}{\Delta t} = \dfrac{d\theta}{dt}$. Average angular velocity $\omega_{av} = \dfrac{\theta_2 - \theta_1}{t_2 - t_1} = \dfrac{\Delta\theta}{\Delta t}$
- Unit : rad/s
- Dimensions : $[M^0L^0T^{-1}]$, which is same as that of frequency.
- Instantaneous angular velocity is a vector quantity, whose direction is normal to the rotational plane and its direction is given by right hand screw rule.
- If ω be the angular velocity, v the linear velocity and r the radius of path, we have the following relation.

$$\vec{v} = \vec{\omega} \times \vec{r}$$

- If n be the frequency then $\omega = 2\pi n$, If T be the time period then $\omega = 2\pi/T$.
- The angular velocity of a rotating rigid body can be either positive or negative, depending on whether it is rotating in the direction of increasing θ (anticlockwise) or decreasing θ (clockwise).
- The magnitude of angular velocity is called the angular speed which is also represented by ω.

Angular Acceleration (α)

- The rate of change of angular velocity is defined as angular acceleration $\vec{\alpha} = \frac{d\vec{\omega}}{dt}$
- Suppose a particle has angular velocity $\vec{\omega}_1$ & $\vec{\omega}_2$ at time t_1 and t_2 respectively

  then average angular acceleration, $\vec{\alpha} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1}$
- It is a vector quantity, whose direction is along the change in direction of angular velocity.
- Unit : rad/s^2
- Dimensions : $M^0L^0T^{-2}$
- Relation between angular acceleration $\vec{\alpha}$ and tangential acceleration $\vec{a}_t$ is $\vec{a}_t = \vec{\alpha} \times \vec{r}$
- Radial or normal acceleration, $\vec{a}_r = \vec{\omega} \times \vec{v}$. Its direction is along the radius.
- Net acceleration $\vec{a} = \vec{a}_t + \vec{a}_r = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$

Comparison of Linear Motion and Rotational Motion

	Linear Motion		Rotational Motion
(i)	If acceleration is 0, v = constant and $s = vt$	(i)	If angular acceleration is 0, ω = constant and $\theta = \omega t$
(ii)	If acceleration a = constant, then	(ii)	If angular acceleration α = constant, then
	(a) $s = \frac{(u+v)}{2}t$		(a) $\theta = \frac{(\omega_0 + \omega)}{2}t$
	(b) $a = \frac{v-u}{t}$		(b) $\alpha = \frac{\omega - \omega_0}{t}$
	(c) $v = u + at$		(c) $\omega = \omega_0 + \alpha t$
	(d) $s = ut + \frac{1}{2}at^2$		(d) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
	(e) $v^2 = u^2 + 2as$		(e) $\omega^2 = \omega_0^2 + 2\alpha\theta$
	(f) $S_{n^{th}} = u + \frac{a}{2}(2n-1)$		(f) $\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(2n-1)$
(iii)	If acceleration is not constant, the above equation will not be applicable. In this case	(iii)	If angular acceleration is not constant, the above equation will not be applicable. In this case
	(a) $v = \frac{ds}{dt}$		(a) $\omega = \frac{d\theta}{dt}$
	(b) $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = v\frac{dv}{ds}$		(b) $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \omega\frac{d\omega}{d\theta}$

GOLDEN KEY POINTS

- In a rigid body, angular velocity of any point w.r.t. any other point is constant and is equal to the angular velocity of the rigid body.

Illustrations

Illustration 1.

A wheel is rotating with angular velocity 2 rad/s. It is subjected to a uniform angular acceleration 2.0 rad/s^2.

(a) What angular velocity does the wheel acquire after 10 s?

(b) How many revolutions will it make in this time interval?

Solution.

The wheel is in uniform angular acceleration, Hence –

$\omega = \omega_o + \alpha t \rightarrow$ Substituting the values of ω_o, α and t, we have

$$\omega = 2 + 2 \times 10 = 22 \text{ rad/s}$$

$\theta = \theta_o + \frac{1}{2}(\omega_o + \omega)t \rightarrow$ Substituting $\theta_o = 0$ for initial position, and ω_o from above equation, we have

$$\theta = 0 + \tfrac{1}{2}(2 + 22)10 = 120 \text{ rad.}$$

In one revolution, the wheel rotates through 2π radians. Therefore the number of complete revolutions n is

$$n = \frac{\theta}{2\pi} = \frac{120}{2\pi} \approx 19$$

Illustration 2.

A flywheel rotates with a uniform angular acceleration. Its angular velocity increases from 20π rad/s to 40π rad/s in 10 seconds. How many rotations did it undergo in this period ? **[AIPMT (Mains) 2006]**

Solution

$\because \omega_2 = \omega_1 + \alpha t \therefore 40\pi = 20\pi + 10\alpha \Rightarrow \alpha = 2\pi$ rad/s^2

angular displacement $\theta = \dfrac{\omega_2^2 - \omega_1^2}{2\alpha} = \dfrac{(40\pi)^2 - (20\pi)^2}{2 \times 2\pi} = \dfrac{1200\pi^2}{4\pi} = 300\pi$ rad

Therefore the number of rotations undergone $= \dfrac{\theta}{2\pi} = \dfrac{300\pi}{2\pi} = 150.$

Illustration 3.

If the position vector $(\vec{r})$ of a point is $(\hat{i} + 2\hat{j} + 3\hat{k})$ m and its angular velocity $(\vec{\omega})$ is $(\hat{i} - \hat{j} + \hat{k})$ rad/s, then find the linear velocity of the particle.

Solution

Linear velocity $\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (-5\hat{i} - 2\hat{j} + 3\hat{k})$ m/s.

Illustration 4.

A particle starts rotating from rest according to the formula $\theta = \frac{3t^3}{20} - \frac{t^2}{3}$ radian. Calculate –

(a) the angular velocity at the end of 5 seconds.

(b) angular acceleration at the end of 5 seconds.

Solution

(a) Angular velocity $\omega = \frac{d\theta}{dt} = \frac{d}{dt}\left[\frac{3t^3}{20} - \frac{t^2}{3}\right] = \frac{3}{20} \times 3t^2 - \frac{1}{3} \times 2t = \frac{9t^2}{20} - \frac{2t}{3}$

Angular velocity at the end of 5 seconds

$$= \frac{9}{20} \times 5 \times 5 - \frac{2}{3} \times 5 = \frac{225}{20} - \frac{10}{3} = 11.25 - 3.33 = 7.92 \text{ rad/s.}$$

(b) Angular acceleration : $\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left[\frac{9t^2}{20} - \frac{2t}{3}\right] = \frac{9}{20} \times 2t - \frac{2}{3} = \frac{9t}{10} - \frac{2}{3}$

Angular acceleration at the end of 5 seconds : $\alpha = \frac{9 \times 5}{10} - \frac{2}{3} = 4.5 - 0.67 = 3.83 \text{ rad/s}^2$.

Illustration 5.

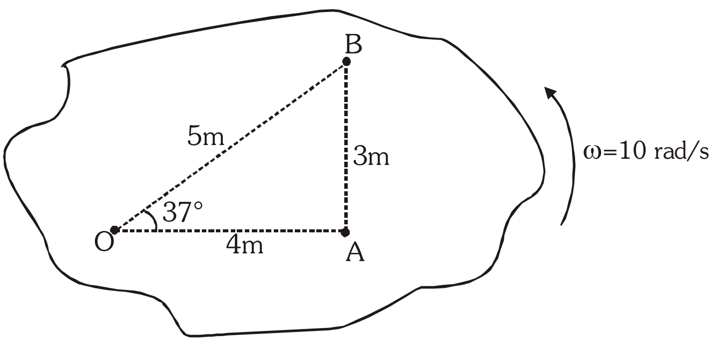
A wheel of perimeter 220 cm rolls on a levelled road at a speed of 9 km/h. How many revolutions does the wheel make per second ?

Solution

$$\text{Frequency } n = \frac{\omega}{2\pi} = \frac{v}{2\pi r} = \frac{9 \times \frac{5}{18}}{2.2} = \frac{25}{22} \text{ rev/s} = 1.136 \text{ rev/s.}$$

Illustration 6.

A rigid lamina is rotating about an axis passing perpendiuclar to its plane through point O as shown in the figure.



The angular velocity of point B w.r.t. A is

(A) 10 rad/s (B) 8 rad/s (C) 6 rad/s (D) 0

Solution **Ans. (A)**

In a rigid body, angular velocity of any point w.r.t. any other point is constant and is equal to the angular velocity of the rigid body.

BEGINNER'S BOX-1

1. A disc rotates about a fixed axis. Its angular velocity ω varies with time according to the equation $\omega = at + b$. Initially at $t = 0$ its angular velocity is 1.0 rad/s and angular position is 2 rad ; at the instant $t = 2$ s, angular velocity is 5.0 rad/s. Determine angular position θ and angular acceleration α when $t = 4$ s.

2. A wheel of radius 1.5 m is rotating at a constant angular acceleration of 10 rad/s^2. Its initial angular speed is $\left(\frac{60}{\pi}\right)$ rpm. What will be its angular speed and angular displacement at $t = 2.0$ s ?

3. A rigid body rotates about a fixed axis with variable angular speed $\omega = A - Bt$ where A and B are constant. Find the angle through which it rotates before it comes to rest.

4. A car has wheels of radius 0.30 m and is travelling at 36 m/s. Calculate :-

(a) the angular speed of the wheel.

(b) If the wheels describe 40 revolutions before coming to rest with a uniform acceleration.

(i) find its angular acceleration and

(ii) the distance covered.

5. A child's top is spun with angular acceleration $\alpha = 4t^3 - 3t^2 + 2t$ where t is in seconds and α is in radians per second-square. At t =0, the top has angular velocity $\omega_0 = 2$ rad/s and a reference line on it is at an angular position $\theta_0 = 1$ rad.

Statement I : Expression for angular velocity $\omega = (2 + t^2 - t^3 + t^4)$ rad/s

Statement II : Expression for angular position $\theta = (1 + 2t - 3t^2 + 4t^3)$ rad

(A) Only statement-I is true

(B) Only statement-II is true

(C) Both of them are true

(D) None of them are true

6. On account of the rotation of earth about its axis :-

(A) the linear velocity of objects at equator is greater than that at other places

(B) the angular velocity of objects at equator is more than that of objects at poles

(C) the linear velocity of objects at all places on the earth is equal, but angular velocity is different

(D) the angular velocity and linear velocity are uniform at all places

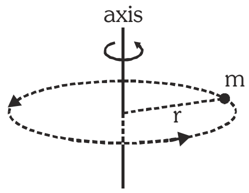
7. A fly wheel originally at rest is to attain an angular velocity of 36 rad/s in 6 s. The total angle it turns through which it turns in the 6 s is :-

(A) 54 radian (B) 108 radian (C) 6 radian (D) 216 radian

8. A rotating rod starts from rest and acquires a rotational speed n = 600 revolutions/minute in 2 seconds with constant angular acceleration. The angular acceleration of the rod is :-

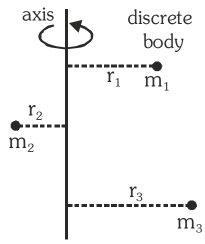
(A) 10π rad/s^2 (B) 5π rad/s^2 (C) 15π rad/s^2 (D) None of these

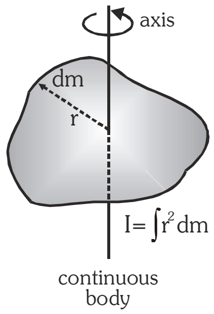
4. MOMENT OF INERTIA



- The measure of the property by virtue of which a body revolving about an axis opposes any change in rotational motion is known as moment of inertia.
- The moment of inertia of a particle with respect to an axis of rotation is equal to the product of its mass and the square of its distance from the rotational axis.

  $I = mr^2$, r = perpendicular distance from the axis of rotation





- Moment of inertia of a system of particles

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ... = \sum mr^2$$

- For a continuous body $I = \int r^2 dm$
- Moment of inertia depends on : (a) mass of the body

  (b) mass distribution of the body

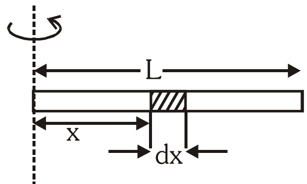
  (c) position of axis of rotation
- Moment of inertia does not depend on :-

  (a) angular velocity (b) angular acceleration (c) torque (d) angular momentum

  UNIT : SI : $kg–m^2$ CGS : $g–cm^2$

  Dimensions : $[M^1 L^2 T^0]$
- As the distance of mass increases from the rotational axis, the moment of inertia (M.I.) increases.

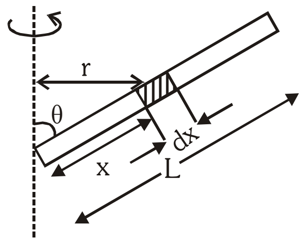
  Moment of Inertia for symmetrical mass distribution
- **Ex.** : Moment of inertia of a rod about an axis passing through its end and perpendicular to length. If the mass of the rod is 'M' & mass of element is dm then



$$dm = \frac{M}{L}dx$$

$$I = \int r^2\,dm = \int_0^L x^2 \frac{M}{L} dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2}{3}.$$

- **Ex.** : Moment of inertia of a rod about an axis inclined at an angle 'θ' with the rod & passing through one end.



$$r = x \sin\theta$$

so $$I = \int r^2 dm = \int_0^L x^2 \sin^2\theta \frac{M}{L} dx$$

$$= \frac{M}{L}\sin^2\theta\left[\frac{x^3}{3}\right]_0^L = \frac{ML^2\sin^2\theta}{3}.$$

5. RADIUS OF GYRATION (K)

It is the distance from the rotation axis where the mass of the object could be assumed to be concentrated without altering the moment of inertia of the body about that axis.

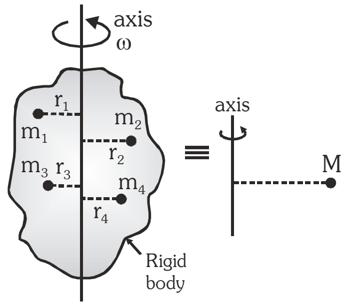
If the mass *m* of the body were actually concentrated at a distance *K* from the axis, the moment of inertia about that axis would be mK^2.

$$K = \sqrt{\frac{I}{m}}$$

K has no meaning without axis of rotation K is a scalar quantity.

The radius of gyration has dimensions of length and is measured in appropriate units of length such as meters.

Ex. :



for the situation shown $K = \sqrt{\dfrac{m_1r_1^2 + m_2r_2^2 + \ldots\ldots + m_nr_n^2}{m_1 + m_2 + \ldots\ldots m_n}}$

if $m_1 = m_2 = m_3 \ldots. = m$ then, $M = mn$ $\quad K = \sqrt{\dfrac{r_1^2 + r_2^2 + \ldots\ldots.r_n^2}{n}}$ $\quad$ n = total number of particles

6. THEOREMS OF MOMENT OF INERTIA

- **Theorem of perpendicular axes (applicable only for two dimensional bodies or plane laminae)**

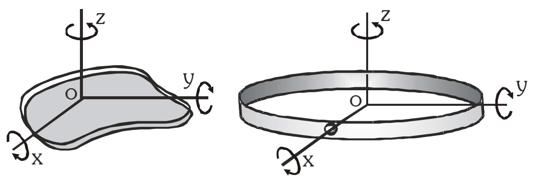
The moment of inertia (MI) of a plane lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about any two mutually perpendicular axes in its own plane intersecting each other at the point through which the perpendicular axis passes.

$$I_z = I_x + I_y$$

where I_x = MI of the body about X–axis

I_y = MI of the body about Y–axis

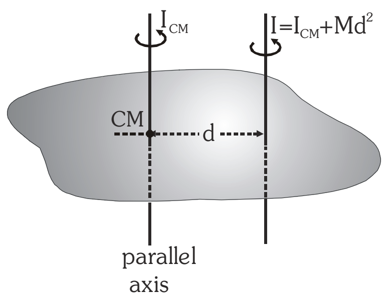
I_z = MI of the body about Z–axis



It is applicable only for two dimensional bodies and cannot be employed for three dimensional bodies.

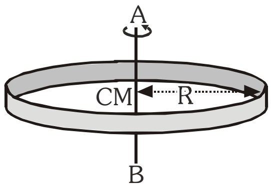
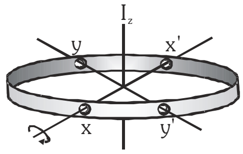
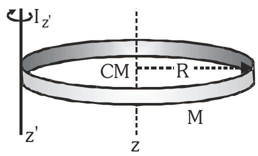
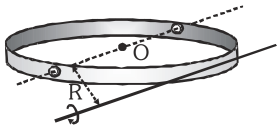
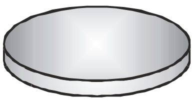
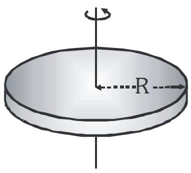
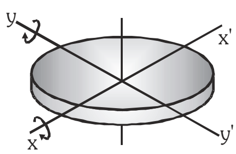
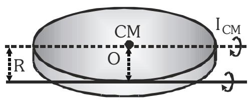
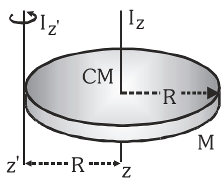
- **Theorem of parallel axes (for all types of bodies)**

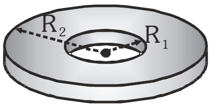
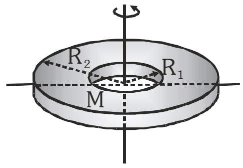
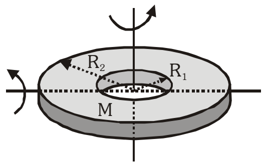
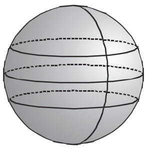
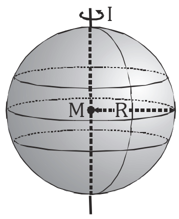
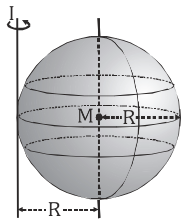
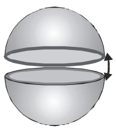
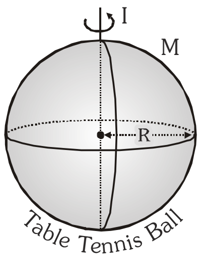
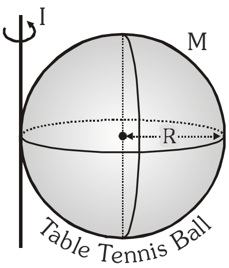
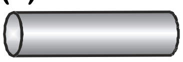
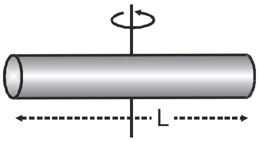
Moment of inertia of a body about any axis is equal to the moment of inertia about a parallel axis passing through the centre of mass plus product of mass of the body and the square of the distance between these two parallel axes. $I = I_{CM} + Md^2$

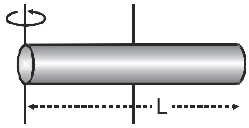
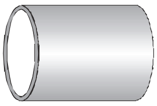
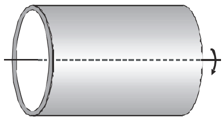
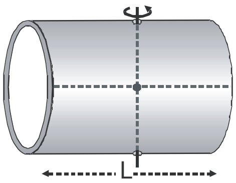
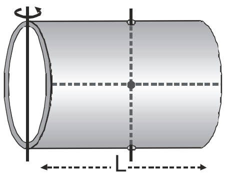
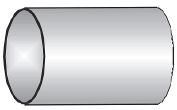
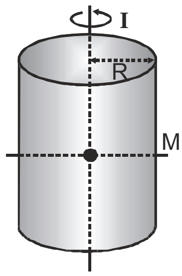
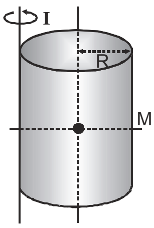
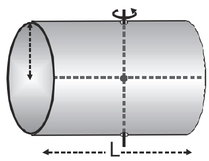


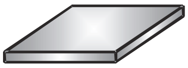
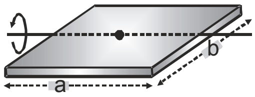
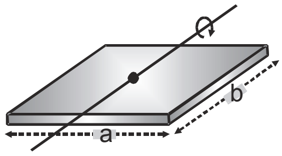
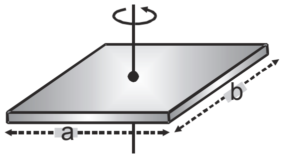
I_{CM} = Moment of inertia about the axis passing through the centre of mass.

7. MOMENT OF INERTIA OF SOME REGULAR BODIES

Type of body		Position of axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(1) Ring  M = Mass R = Radius	(a)	About an axis perpendicular to its plane and passing through the centre		MR^2	R
	(b)	About the diametric axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(c)	About an axis tangential to the ring and perpendicular to its plane		$2MR^2$	$\sqrt{2}R$
	(d)	About an axis tangential to the ring and lying in its plane		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
(2) Disc  M = Mass R = Radius	(a)	About an axis passing through the centre and perpendicular to its plane		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(b)	About a diametric axis		$\frac{MR^2}{4}$	$\frac{R}{2}$
	(c)	About an axis tangential to the disc and lying in its plane		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$
	(d)	About an axis tangential to the disc and perpendicular to its plane		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$

Type of body	Position of axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(3) Annular disc 	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{M}{2}\left[R_1^2+R_2^2\right]$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
M = Mass R_1 = Inner Radius R_2 = Outer Radius	(b) About a diametric axis		$\frac{M}{4}\left[R_1^2+R_2^2\right]$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$
(4) Solid Sphere 	(a) About its diametric axis		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
M = Mass R = Radius	(b) About a tangent to the Sphere		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$
(5) Thin Hollow Sphere 	(a) About its diametric axis		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
(Thin spherical shell) M = Mass R = Radius (Thickness negligible)	(b) About a tangent to the sphere		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$
(6) Thin Rod  **Thickness is negligible w.r.t. length**	(a) About an axis passing through the centre of mass and perpendicular to its length		$\frac{ML^2}{12}$	$\frac{L}{\sqrt{12}}$

Type of body	Position of axis of rotation		Figure	Moment of Inertia (I)	Radius of gyration (K)
M = Mass L = Length	(b)	About an axis passing through one end and perpendicular to its length		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
(7) Thin Hollow Cylinder  M = Mass R = Radius L = Length	(a)	About its geometrical axis which is parallel to its length		MR^2	R
	(b)	About an axis which is perpendicular to its length and passes through its centre of mass		$\frac{MR^2}{2}+\frac{ML^2}{12}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{12}}$
	(c)	About an axis perpendicular to its length and passing through one end of the cylinder		$\frac{MR^2}{2}+\frac{ML^2}{3}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{3}}$
(8) Solid Cylinder M = Mass R = Radius L = Length 	(a)	About its geometrical axis, which is parallel to its length		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	(b)	About an axis tangential to the cylindrical surface and parallel to its geometrical axis		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
	(c)	About an axis passing through the centre of mass and perpendicular to its length		$\frac{ML^2}{12}+\frac{MR^2}{4}$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{4}}$

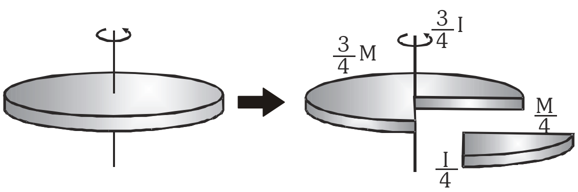
Type of body	Position of axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(9) Rectangular Plate  M = Mass a = Length b = Breadth	(a) About an axis passing through its centre of mass and perpendicular to side b in its plane		$\frac{Mb^2}{12}$	$\frac{b}{2\sqrt{3}}$
	(b) About an axis passing through its centre of mass and perpendicular to side a in its plane.		$\frac{Ma^2}{12}$	$\frac{a}{2\sqrt{3}}$
	(c) About an axis passing through the centre of mass and perpendicular to its plane		$\frac{M\left(a^2+b^2\right)}{12}$	$\sqrt{\frac{a^2+b^2}{12}}$

GOLDEN KEY POINTS

- Moment of inertia is not a vector as direction CW or ACW is not to be specified and also not a scalar as it has different values in different directions (i.e. about different axes). It is a tensor quantity.
- If a wheel is to be made by using two different materials then for larger moment of inertia, larger density material should lie in the periphery .
- Radius of gyration depends on

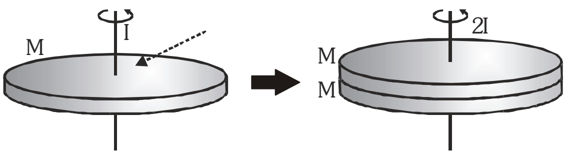
  (i) Axis of rotation (ii) Distribution of mass of body
- Radius of gyration does not depend on

  (i) Mass of the body (ii) Angular quantities (angular displacement, angular velocity etc.)
- For Symmetrical separation radius of gyration remains unchanged.



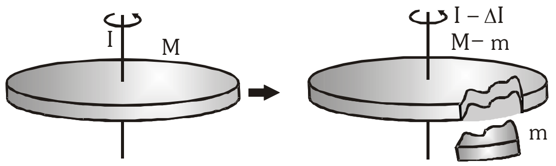
$\Rightarrow$ K remains unchanged

- For Symmetrical attachment radius of gyration remains unchanged.



$\Rightarrow$ K remains unchanged

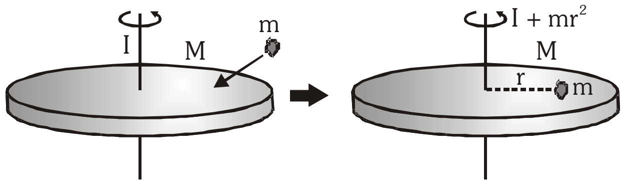
- For asymmetrical separation radius of gyration will change.



$\Rightarrow \quad K = \sqrt{\dfrac{I - \Delta I}{M - m}} \qquad \Rightarrow$ K change

Here ΔI = M. I. of detached mass w.r.t. same axis.

- For asymmetrical attachment radius of gyration will be changed.

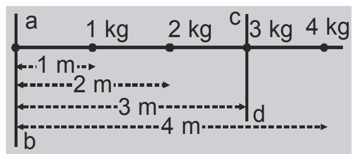


$\Rightarrow \qquad K = \sqrt{\dfrac{I + mr^2}{M + m}} \qquad \Rightarrow$ K change

Illustrations

Illustration 7.

Calculate the M.I. of the given particles system about axis a - b and c - d



Solution.

$I_{a-b} = 1 \times (1)^2 + 2 \times (2)^2 + 3 \times (3)^2 + 4 \times (4)^2 = 100$ kg-m²

$I_{c-d} = 1 \times (2)^2 + 2 \times (1)^2 + 3 \times (0)^2 + 4 \times (1)^2 = 10$ kg-m²

Illustration 8.

Calculate the moment of inertia about the rotational axis XX' in following figures.

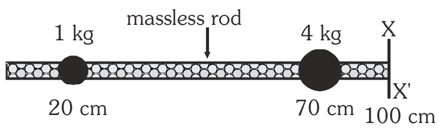


fig (a)

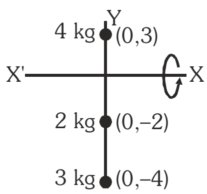


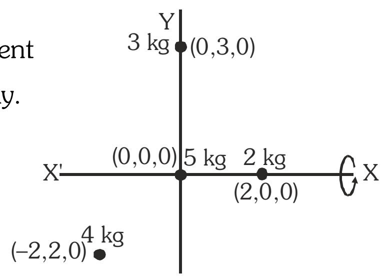
fig (b) (not to scale)

Solution. (a) $I_{xx'} = 4 \times (0.3)^2 + 1 \times (0.8)^2 = 1$ kg–m²

(b) $I_{xx'} = 4 \times (3)^2 + 2 \times (2)^2 + 3 \times (4)^2 = 92$ kg–m²

Illustration 9.

Four bodies of masses 5 kg , 2 kg, 3 kg, and 4 kg are respectively placed at positions (0,0,0), (2,0,0), (0, 3, 0) and (–2, –2, 0). Calculate the moment of inertia of the system of bodies about x– axis, y–axis and z–axis respectively.



Solution.

$I_x = 3 \times (3)^2 + 4 \times (2)^2 = 43$ units

$I_y = 2 \times (2)^2 + 4 \times (2)^2 = 24$ units

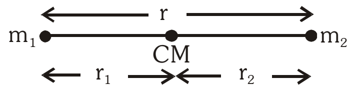
$I_z = 2 \times (2)^2 + 3 \times (3)^2 + 4 \times (2\sqrt{2})^2 = 8 + 27 + 32$

$= 67$ units $(I_z = I_x + I_y = 43 + 24 = 67)$

Illustration 10.

Two masses m_1 and m_2 are placed at a separation r. Find out the moment of inertia of the system about an axis passing through its centre of mass and perpendicular to the line joining the masses.

Solution.



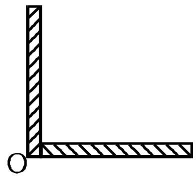
$m_1r_1 = m_2r_2$ and $r_1 + r_2 = r \Rightarrow r_1 = \dfrac{m_2 r}{m_1+m_2}$, $r_2 = \dfrac{m_1 r}{m_1+m_2}$

Moment of inertia $I = m_1r_1^2 + m_2r_2^2 = m_1\left(\dfrac{m_2r}{m_1+m_2}\right)^2 + m_2\left(\dfrac{m_1r}{m_1+m_2}\right)^2 = \left(\dfrac{m_1m_2}{m_1+m_2}\right)r^2$

Note : Here $I = \mu r^2$ where μ (reduced mass) $= \dfrac{m_1m_2}{m_1+m_2}$.

Illustration 11.

Two identical rods each of mass M and length L are kept according to figure. Find the moment of inertia of rods about an axis passing through O and perpendicular to the plane of rods. **[AIPMT (Mains) 2004]**



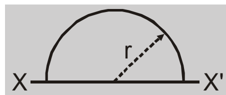
Solution

$\because$ Moment of inertia of each rod about an axis passing through an end $= \dfrac{ML^2}{3}$

$$\therefore\ I_{\text{given system}} = \frac{ML^2}{3} + \frac{ML^2}{3} = \frac{2ML^2}{3}.$$

Illustration 12.

A uniform wire of length ℓ and mass M is bent in the shape of a semicircle of radius r as shown in figure. Calculate moment of inertia about the axis XX'



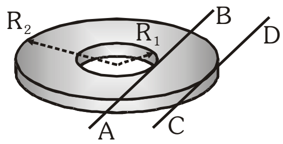
Solution.

Length of the wire $\ell = \pi r \quad \Rightarrow \quad r = \dfrac{\ell}{\pi} \quad \Rightarrow \quad I_{xx'} = \dfrac{Mr^2}{2} = \dfrac{M\ell^2}{2\pi^2}$.

Illustration 13.

Calculate the moment of inertia of an annular disc about an axis which lies in the plane of the disc and tangential to the (i) inner circle and (ii) outer circle. Mass of the disc is M and its inner radius is R_1 and outer radius is R_2.

Solution.

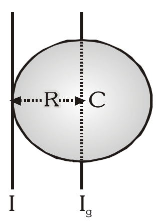


(i) M.I. about an axis tangential to the inner circle is $I_{AB} = \dfrac{M}{4}(R_1^2+R_2^2) + MR_1^2$

(ii) M.I. about an axis tangential to the outer circle is $I_{CD} = \dfrac{M}{4}(R_1^2+R_2^2) + MR_2^2$

Illustration 14.

The moment of inertia of a sphere is 40 kg–m² about its diametric axis. Determine the moment of inertia about any tangent.



Solution.

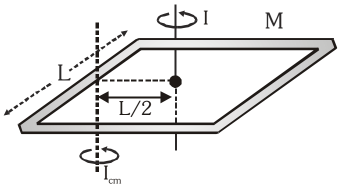
Given that $\dfrac{2}{5}MR^2 = 40 \quad \Rightarrow MR^2 = 100$ kg-m².

By theorem of parallel axes,

$I = I_{CM} + MR^2 = \dfrac{2}{5}MR^2 + MR^2 = \dfrac{7}{5}MR^2 = \dfrac{7}{5}\times 100 = 140$ kg–m².

Illustration 15.

Four rods are arranged in the form of a square. Calculate the moment of inertia of the system of rods about an axis passing through the centre and perpendicular to the plane of the square (Assume that each rod has a mass M and length L)



Solution.

From parallel axes theorem,

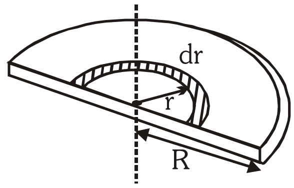
$$I = 4I_{CM} + 4M\left(\frac{L}{2}\right)^2 = 4\left(\frac{ML^2}{12}\right) + 4\frac{ML^2}{4} = \frac{4}{3}ML^2.$$

Illustration 16.

Calculate the moment of Inertia of a semicircular disc of mass M and radius R about an axis passing through its centre and perpendicular to its plane.

Solution

Let us assume a ring of radius 'r' & thickness 'dr'



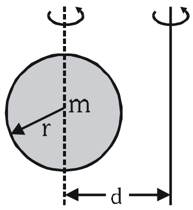
$$dm = \frac{M}{\frac{\pi R^2}{2}}(\pi r dr) = \frac{2Mrdr}{R^2}$$

$$I = \int r^2 dm = \int_0^R r^2 \frac{2Mr}{R^2} dr = \frac{2M}{R^2}\int_0^R r^3 dr = \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_0^R \Rightarrow I = \frac{MR^2}{2}$$

Illustration 17.

The radius of gyration of a solid sphere of radius r about a certain axis is r. Calculate the distance of that axis from the centre of the sphere.

Solution.



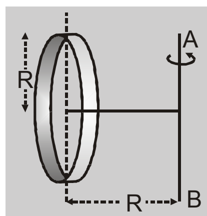
From paralle axes theorem,

$$\because \quad I = I_{CM} + md^2 \quad \therefore\ mr^2 = \frac{2}{5}mr^2 + md^2$$

$$\Rightarrow \quad d = \sqrt{\frac{3}{5}}r = \sqrt{0.6}\ r.$$

Illustration 18.

Find the moment of inertia of the ring shown in figure about the axis AB.



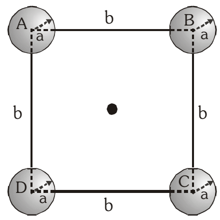
Solution.

From parallel axes theorem,

$$I_{AB} = I_{CM} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$

Illustration 19.

Four spheres each of diameter 2a and mass M are placed with their centres lying on the four corners of a square of side b. Calculate the moment of inertia of the system about one side of the square taken as the axis.



Solution.

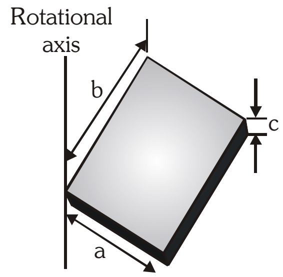
ABCD is square of side b and four spheres each of mass M and radius a are placed at the four corners. Suppose we have to calculate the moment of inertia of the system about the side BC as the axis. Therefore the moment of inertia of the system (M.I.).

= M.I. of sphere at A about BC + MI of sphere at B about BC + MI of sphere at C about BC + MI of sphere at D about BC $= \left(\frac{2}{5}Ma^2 + Mb^2\right) + \frac{2}{5}Ma^2 + \frac{2}{5}Ma^2 + \left(\frac{2}{5}Ma^2 + Mb^2\right) = \frac{8}{5}Ma^2 + 2Mb^2 = \frac{2}{5}M(4a^2 + 5b^2)$.

Illustration 20.

The uniform solid block shown in figure has mass M and edge dimensions a, b, and c. Calculate its rotational inertia about an axis passing through one corner and perpendicular to the large faces.

Solution.



Use the parallel - axes theorem. The rotational inertia of a rectangular slab about an axis through the centre and perpendicular to the large face is given by

$$I_{cm} = \frac{M}{12}\left(a^2+b^2\right)$$

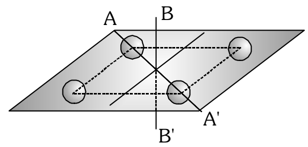
A parallel axis through a corner is at distance $h = \sqrt{(a/2)^2+(b/2)^2}$

from the centre, so $I = I_{cm} + Mh^2 = \frac{M}{12}(a^2 + b^2) + \frac{M}{4}(a^2 + b^2) = \frac{M}{3}(a^2 + b^2)$.

Illustration 21.

Four spheres each of mass M and diameter 2a are placed at the corners of square of side b as shown below. Calculate the moment of inertia about (a) axis BB' (b) axis AA'

Solution.



(a) Axis BB' passes through the centre and perpendicular to the plane

$$I_{BB'} = 4 \times I_{CM} + 4M\left[\frac{b}{\sqrt{2}}\right]^2 = 4 \times \frac{2}{5}Ma^2 + 4M \times \frac{b^2}{2} = 4M\left[\frac{2}{5}a^2+\frac{b^2}{2}\right]$$

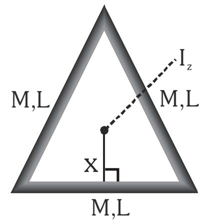
(b) Axis AA' passes through the centres of the spheres

$$I_{AA'} = 4I_{CM} + 2M\left[\frac{b}{\sqrt{2}}\right]^2 = 4 \times \frac{2}{5}Ma^2 + 2M\frac{b^2}{2} = \frac{8}{5}Ma^2+Mb^2.$$

Illustration 22.

Three rods are arranged in the form of an equilateral triangle. Calculate the M.I. about an axis passing through the geometrical centre and perpendicular to the plane of the triangle (Assume that mass and length of each rod is M and L respectively).

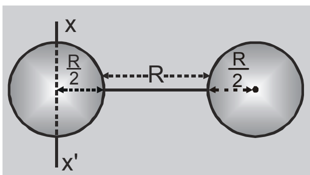
Solution.



$$I = 3I_{CM}+3M\,x^2 = \frac{3ML^2}{12} + 3M\left(\frac{L}{2\sqrt{3}}\right)^2 = \frac{ML^2}{4} + \frac{ML^2}{4} = \frac{ML^2}{2}.$$

Illustration 23.

Diameter of each spherical shell is R and mass M they are joined by a light and massless rod. Calculate the moment of inertia of the system about xx' axis.

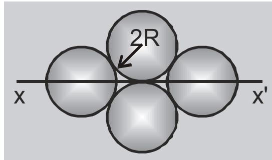


Solution.

$$I_{system} = \frac{2}{3}M\left(\frac{R}{2}\right)^2 + \left[\frac{2}{3}M\left(\frac{R}{2}\right)^2 + M(2R)^2\right] = \frac{1}{3}MR^2 + 4MR^2 = \frac{13}{3}MR^2$$

Illustration 24.

The moment of inertia of a sphere about its diameter is I. Four such spheres are arranged as shown in figure. Find the moment of inertia of the system about the axis XX'. (radius of each sphere is 2R).



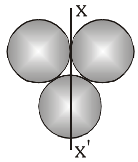
Solution.

Unless explicitly mentioned any sphere should be assumed to be solid.

$$I = \frac{2}{5}M(2R)^2 \text{ or, } M(2R)^2 = \frac{5}{2}I \quad \Rightarrow \quad I_{system} = I + 2[I + M(2R)^2] + I = I + 2I + 2\times\frac{5}{2}I + I = 9I$$

Illustration 25.

Adjoining diagram shows three rings, each of which has a mass M and radius R. Find the moment of inertia of this system about the axis XX'



Solution.

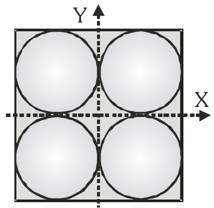
$$I_{system} = 2 \times I_{upper} + I_{lower} = 2\times\frac{3}{2}MR^2 + \frac{1}{2}MR^2 = \frac{7}{2}MR^2.$$

Illustration 26.

Four holes of radius R are cut from a thin square plate of side 4R and mass M. Determine the moment of inertia of the remaining portion about Z–axis.

Solution.

M = Mass of the square plate before the holes were cut.



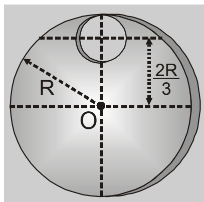
Mass of each hole $m = \left[\frac{M}{16R^2}\right]\pi R^2 = \frac{\pi M}{16}$.

$\therefore$ Moment of inertia of the remaining portion,

$$I = I_{square} - 4I_{hole} = \frac{M}{12}(16R^2 + 16R^2) - 4\left[\frac{mR^2}{2} + m(\sqrt{2}R)^2\right] = \frac{8}{3}MR^2 - 10mR^2 = \left[\frac{8}{3} - \frac{10\pi}{16}\right]MR^2.$$

Illustration 27.

A thin uniform disc has a mass 9M and radius R. A disc of radius $\frac{R}{3}$ is cutoff as shown in figure. Find the moment of inertia of the remaining disc about an axis passing through O and perpendicular to the plane of disc.



Solution.

As the mass is uniformly distributed on the disc,

so mass density (per unit area) $= \frac{9M}{\pi R^2}$. Mass of removed portion $= \frac{9M\pi}{\pi R^2} \times \left[\frac{R}{3}\right]^2 = M$

So the moment of inertia of the removed portion about the stated axis by theorem of parallel axes is :

$$I_1 = \frac{M}{2}\left[\frac{R}{3}\right]^2 + M\left[\frac{2R}{3}\right]^2 = \frac{MR^2}{2} \qquad \text{......(i)}$$

The moment of inertia of the original complete disc about the stated axis is I_2 then $I_2 = 9M\frac{R^2}{2}$ .....(ii)

So the moment of inertia of the left over disc shown in fig. is $I_2 - I_1$.

i.e., $\quad I_2 - I_1 = 4MR^2$.

BEGINNER'S BOX-2

1. Point masses of 1, 2, 3 and 10 kg are lying at the points (0, 0), (2m,0) (0, 3m) and (– 2m, – 2m) respectively in x–y plane. Find the moment of inertia of this system about y - axis. (in kg–m²).

2. The moment of inertia of a ring and a disc of same mass about their diameters are same. What will be the ratio of their radii ?

3. Two rings have their moments of inertia in the ratio 4 : 1 and their diameters are in the ratio 4 : 1. Find the ratio of their masses.

4. Two bodies of masses 1 kg and 2 kg are attached to the ends of a 2 m long weightless rod. This system is rotating about an axis passing through the middle point of rod and perpendicular to length. Calculate the M.I. of system.

5. If the moment of inertia of a disc about an axis tangential and parallel to its surface be I, then what will be the moment of inertia about an axis tangential but perpendicular to the surface ?

6. A disc is recast into a thin hollow cylinder of same radius. Which will have larger moment of inertia ? Explain.

7. Why spokes are fitted in a cycle wheel ?

8. A circular disc A of radius r is made from an iron plate of thickness t and another circular disc B of radius 4r is made from an iron plate of thickness t/4. Find the relation between the moment of inertia I_A and I_B.

9. A square plate of side ℓ has mass per unit area μ. Find its moment of inertia about an axis passing through the centre and perpendicular to its plane.

10. Three masses m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of side a. What is the moment of inertia of the system about an axis along the median passing through m_1 ?

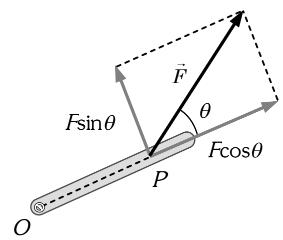
11. Two rings having the same radius and mass are placed such that their centres are at a common point and their planes are perpendicular to each other. Find the moment of inertia of the system about an axis passing through the centre and perpendicular to the plane of one of the rings. (M = mass of each ring and R = radius).

12. Moment of inertia of a sphere about its diameter is 2/5 MR². What is its moment of inertia about an axis perpendicular to its two diameters and passing through their point of intersection ?

8. TORQUE

Torque is the rotational analogue of force and expresses the tendency of a force applied to an object to cause rotation in it about a given point.

Consider a rod pivoted at the point O. A force $\vec{F}$ is applied on it at the point P. The component F cosθ of the force along the rod is counterbalanced by the reaction force of the pivot and cannot contribute in rotating the rod. It is the component F sinθ of the force perpendicular to the rod, which is responsible for rotation of the rod. Moreover, farther is the point P from O, where the force is applied easier it is to rotate the rod.

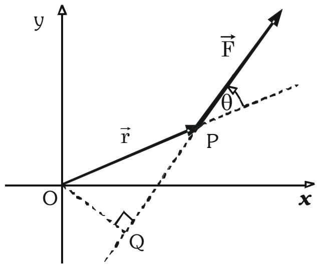


This explains why handle on a door is attached as far as possible away from the hinges.

Magnitude of torque of a force is proportional to the product of distance of point of application of the force from the pivot and the magnitude of the perpendicular component F sinθ of the force. Denoting torque by symbol τ, the distance of point of application of force from the pivot by r, we can write

$$\tau_0 \propto rF\sin\theta$$

Since rotation has sense of direction, torque should also be a vector. Its direction is given by the right hand rule. Now we can express torque by the cross product of $\vec{r}$ and $\vec{F}$.



$$\vec{\tau}_o = \vec{r} \times \vec{F} \qquad [1]$$

The magnitude of torque

Torque = Force × perpendicular distance of line of action of force from the axis of rotation.

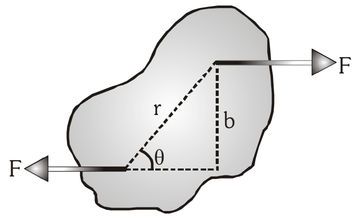
$$\tau = r\ F\ \sin\theta$$

so Moment of force is known as torque

- **Unit of torque :** N–m (same as that of work or energy)
- **Dimensions of torque :** $[ML^2T^{-2}]$
- Unit of torque (N-m) cannot be written as joule, because joule is used specifically for work or energy.
- **Couple of forces**

  When two forces of equal magnitude act on different points and in opposite directions with distinct lines of action these force form a couple. This couple tries to rotate body. Moment of couple

  $\tau = Fr\sin\theta = Fb$



- Rotation of a door about a hinge, rotation of grinding wheel about a pivot or unbolting a nut by a pipe– wrench can be cited as examples involving torque . These examples represent rotational effects.

$$\tau = \text{constant} \Rightarrow Fr\sin\theta = \text{constant} \Rightarrow F = \frac{\text{constant}}{r\sin\theta}$$

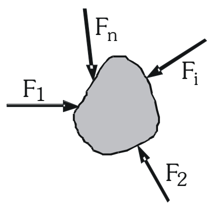
Longer the arm and greater the value of $\sin\theta$, lesser will be the force required for producing desired rotational effect. This fact explains is why, it is much easier to rotate a body about a given axis when the force is applied at maximum distance from the axis of rotation and normal to the arm.

9. ROTATIONAL EQUILIBRIUM

A rigid body is said to be in a state of rotational equilibrium if its angular acceleration is zero. Therefore a body in rotational equilibrium must either be at rest or in rotation with a constant angular velocity.

If a rigid body is in rotational equilibrium under the action of several coplanar forces, the resultant torque of all the forces about any axis perpendicular to the plane containing the forces must be zero.

In the figure a body is shown under the action of several external coplanar forces F_1, F_2, ...F_i, and F_n.



$$\Sigma\vec{\tau}_P = 0$$

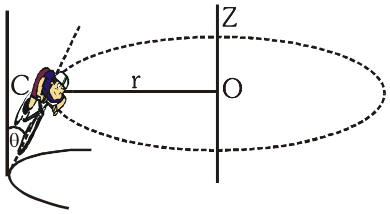
Here P is a point in the plane of the forces about which we calculate torque of all the external forces acting on the body.

Note : (i) A body cannot be in rotational equilibrium under the action of a single force unless the line of action passes through the axis of rotation.

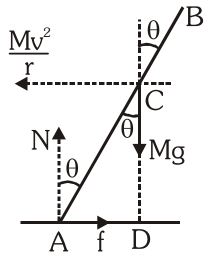
(ii) If a body is in rotational equilibrium under the action of three forces, the lines of action of the three forces must be either concurrent or parallel.

10. BENDING OF A CYCLIST ON A HORIZONTAL TURN

Suppose a cyclist is moving at a speed υ on a circular horizontal road of radius r. Consider the cycle and the rider together as the system. The centre of mass C (figure a) of the system is going in a circle with the centre at O and radius r.



(a)



(b)

As seen from a frame rotating with the same angular velocity as the system is in equilibrium

So $F_{net} = 0$ and $\tau_{net} = 0$

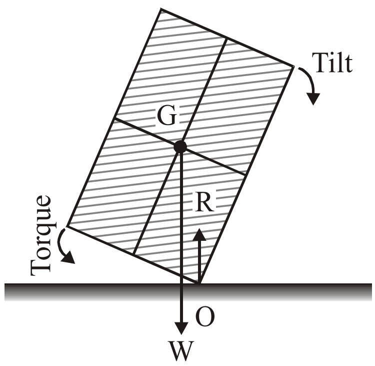
for rotational equilibrium

$$\tau_A = 0 \quad \Rightarrow \quad Mg\,(AD) = \frac{M\upsilon^2}{r}(CD) \quad \Rightarrow \quad \frac{AD}{CD} = \frac{\upsilon^2}{rg} \quad \Rightarrow \quad \tan\theta = \frac{\upsilon^2}{rg}$$

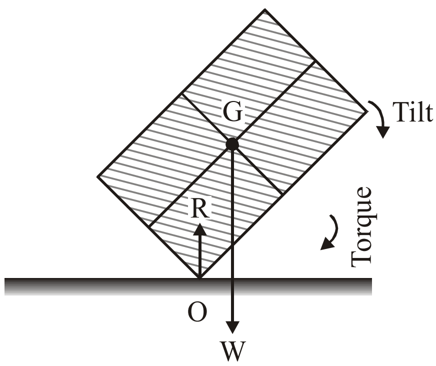
Thus the cyclist bends at an angle $\tan^{-1}\left(\frac{\upsilon^2}{rg}\right)$ with the vertical.

GOLDEN KEY POINTS

- Torque is an axial vector, i.e. Its direction is always perpendicular to the plane containing vector $\vec{r}$ and $\vec{F}$ and its direction is determined by the right hand screw rule.
- Generally positive sign is given to all torques acting to turn a body anticlockwise and a minus to all torques tending to turn it clock wise.
- Wrench with longer handle is more useful than one with shorter handle
- A rigid body is said to be in equilibrium, if it is in translational as well as rotational equilibrium. To analyze such problems conditions for both the equilibriums must be applied.



(A)

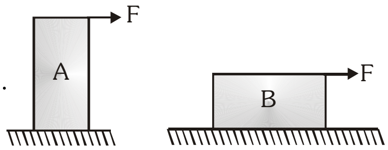


(B)

On tilting, a body will restore its initial position due to torque of weight about the point O till the line of action of weight passes through O. Upon on tilting it further a the body topples due to torque of weight about O; the line of action of action of weight does not pass through the base.

Toppling :

For the shown situations (A) and (B), more chances of toppling is for (A).



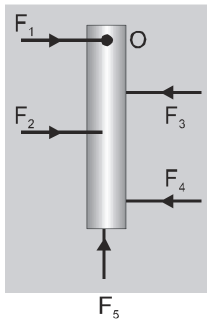
In case of toppling, normal reaction must pass through the end point about which the body topples. .

Illustrations

Illustration 28.

Different forces are applied on a pivoted scale as shown. Which of the following forces will produce torque ?

Solution.



$F_1 \neq 0, \ \ \tau_1 = 0$

(As line of action of force passes from the pivoted point O)

$F_2 \neq 0$ and produces τ_2 (Anticlock wise)

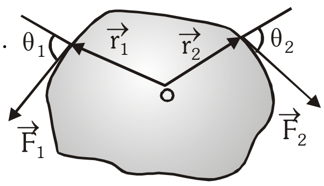
$F_3 \neq 0$ produces τ_3 (Clockwise)

$F_4 \neq 0$ Produces τ_4 (Clockwise)

$F_5 \neq 0, \tau_5 = 0$; as line of action of force passes from the pivoted point O.

Illustration 29.

The body in shown figure is pivoted at O, and two forces act on it as shown.



(a) Find an expression for the net torque on the body about the pivot.

(b) If $r_1 = 1.30$ m, $r_2 = 2.15$ m, $F_1 = 4.20$ N, $F_2 = 4.90$ N, $\theta_1 = 75^0$ and $\theta_2 = 60^0$, what is the net torque about the pivot?

Solution

(a) Take a torque that tends to cause a counter clockwise rotation to be positive and a torque that tends to cause a clockwise rotation to be negative. Thus a positive torque of magnitude $r_1F_1\sin\theta_1$ is associated with $\vec{F}_1$ and a negative torque of magnitude $r_2F_2\sin\theta_2$ is associated with $\vec{F}_2$. Both of these are about O. The net torque about O is $\tau = r_1F_1\sin\theta_1 - r_2F_2\sin\theta_2$

(b) Substitute the given values to obtain τ = (1.30 m) (4.20 N) sin 75° – (2.15 m) (4.90 N) sin 60°

$$= -3.85 \text{ N-m.}$$

Illustration 30.

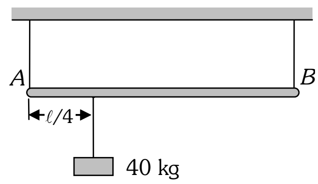
A force $10\ (-\hat{k})$ N acts on the origin of the coordinate system. Find the torque about the point (1m, –1m, 0).

Solution.

Torque $\vec{\tau} = \vec{r} \times \vec{F} = \left[(0-1)\hat{i} + (0+1)\hat{j} + (0-0)\hat{k}\right] \times (-10)\hat{k} = 10(-\hat{i} - \hat{j}) = -10(\hat{i} + \hat{j})$ N-m

Illustration 31.

A uniform rod of 20 kg is hanging in a horizontal position with the help of two threads. It also supports a 40 kg mass as shown in the figure. Find the tensions developed in each thread.

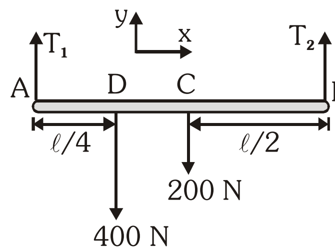


Solution.

Free body diagram of the rod is shown in the figure.

Translational equilibrium requires

$\Sigma F_y = 0 \Rightarrow$ $T_1 + T_2 = 400 + 200 = 600$ N (i)



Rotational equilibrium: Applying the condition about A, we get T_2.

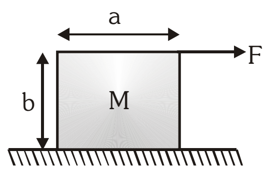
$\Sigma \vec{\tau}_A = \vec{0} \Rightarrow$ $-400(\ell/4) - 200(\ell/2) + T_2\ell = 0$

$T_2 = 200$ N

From equation (i)

$T_1 = 400$ N.

Illustration 32.

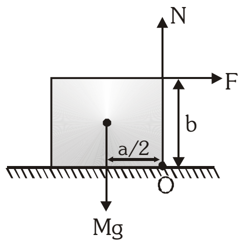


Find the minimum value of F for the block to topple about an edge.

Solution.

When the block is about to topple the normal reaction N shifts to the edge through O.

FBD during toppling

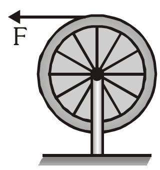


Taking torque about O

$F\ (b) = Mg\left(\frac{a}{2}\right) \Rightarrow F_{min} = \frac{Mga}{2b}$.

Illustration 33.

A string is wrapped around the rim of a wheel of moment of inertia 0.20 kg–m² and radius 20 cm. The wheel is free to rotate about its axis. Initially the wheel is at rest. The string is now pulled by a force of F = 20 N. Find the angular velocity of the wheel after 5 s.

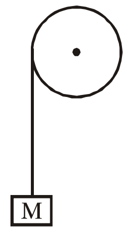


Solution

Angular impulse = change in angular momentum

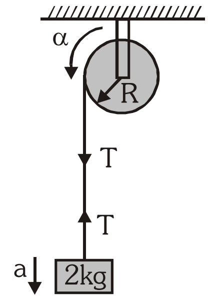
$(\tau t = I\omega - 0) \Rightarrow \omega = \frac{\tau t}{I} = \frac{FRt}{I} = \frac{20 \times 0.2 \times 5}{0.2} = 100$ rad/s

Illustration 34.



A fixed pulley of radius 20 cm and moment of inertia 0.32 kg·m^2 about its axle has a massless cord wrapped around its rim. A 2 kg mass M is attached to the end of the cord. The pulley can rotate about its axis without any friction. Find the acceleration of the mass M. (Assume g = 10 m/s^2) **[AIPMT (Mains) 2007]**

Solution.



For the motion of the block $2g - T = 2a$

For the motion of the pulley $\tau = TR = I\alpha$

$$\because a = \alpha R \quad \therefore T = \frac{Ia}{R^2} \Rightarrow 2g - \frac{Ia}{R^2} = 2a \Rightarrow a = \frac{g}{1+\frac{I}{2R^2}}$$

$$\Rightarrow a = \frac{10}{1+\frac{0.32}{2\times0.2\times0.2}} = \frac{10}{1+4}\ \frac{10}{5} = 2\ \text{m/s}^2.$$

BEGINNER'S BOX-3

1. A force $\vec{F} = (\hat{i} - 2\hat{j} + 5\hat{k})$ N is acting at a point $\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k})$ m. Find torque about the origin.

2. The density of a rod AB increases continuously from A to B. Is it easier to set it in rotation by clamping it at A and applying a perpendicular force at B or by clamping it at B and applying the force at A ? Explain your answer.

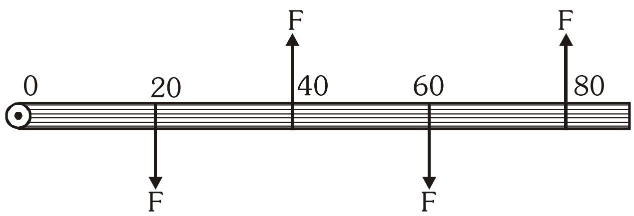
3. A uniform disc of radius 20 cm and mass 2 kg can rotate about a fixed axis through the centre and perpendicular to its plane. A massless cord is round along the rim of the disc. If a uniform force of 2 newtons is applied on the cord, tangential acceleration of a point on the rim of the disc will be :-

(A) 1 m/s^2 (B) 2 m/s^2 (C) 3.2 m/s^2 (D) 1.6 m/s^2

4. A door 1.6 m wide requires a force of 1 N to be applied at the free end to open or close it. The force that is required to be applied at a point 0.4 m distant from the hinges for opening or closing the door is :-

(A) 1.2 N (B) 3.6 N (C) 2.4 N (D) 4 N

5. Four equal and parallel forces are acting on a rod (as shown in figure at distances of 20 cm, 40 cm, 60 cm and 80 cm respectively from one end of the rod. Under the influence of these forces the rod



(A) is at rest

(B) experiences a torque

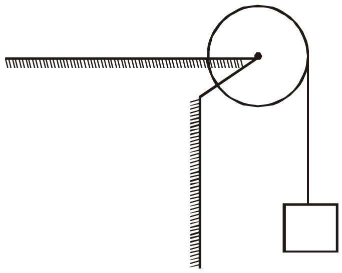
(C) experiences a linear motion

(D) experiences a torque and also a linear motion

6. A weightless rod is acted upon by upward parallel forces of 2 N and 4 N at ends A and B respectively. The total length of the rod AB = 3 m. To keep the rod in equilibrium a force of 6 N should act in the following manner:

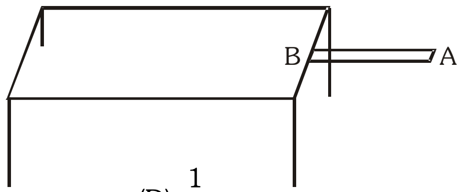
(A) downward at any point between A and B
(B) downward at the midpoint of AB
(C) downward at a point C such that AC = 1 m
(D) downward at a point D such that BD = 1 m

7. Figure shows a uniform disc, with mass M = 2.4 kg and radius R = 20 cm, mounted on a fixed horizontal axle. A block of mass m = 1.2 kg hangs from a massless cord which is wrapped around the rim of the disc. The tension in the cord is :



(A) 12 N
(B) 20 N
(C) 24 N
(D) 6 N

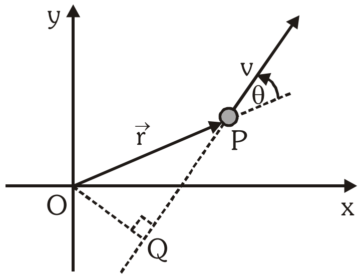
8. A thin uniform stick of length ℓ and mass m is held horizontally with its end B hinged on the edge of a table. Point A is suddenly released. The acceleration of the centre of mass of the stick at the time of release, is :- **[AIPMT (Mains) 2007]**



(A) $\frac{3}{4}g$ (B) $\frac{3}{7}g$ (C) $\frac{2}{7}g$ (D) $\frac{1}{7}g$

11. ANGULAR MOMENTUM

Angular momentum of a particle



Angular momentum $\vec{L}_o$ about the origin O of a particle of mass m moving with velocity $\vec{v}$ is defined as the moment of its linear momentum $\vec{p} = m\vec{v}$ about the point O.

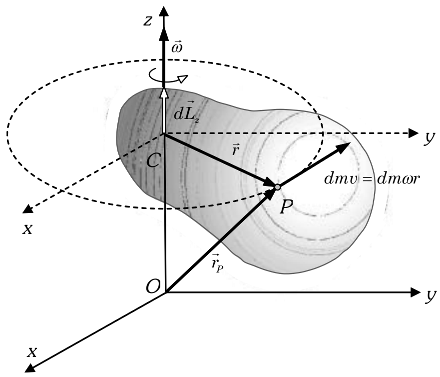
$$\vec{L}_o = \vec{r} \times (m\vec{v})$$

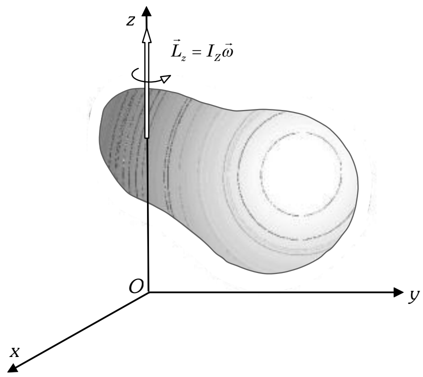
Angular Momentum of a Rigid Body

Angular momentum is a measure of the quantity of rotational motion in a body. The angular momentum of a system of particles is the sum of the angular momenta all the particles forming the system. A rigid body is an assemblage of large number of particles maintaining their mutual distances intact under all circumstances.

Angular Momentum about an axis

The following figure shows a particle P of a rigid body rotating about the z-axis angular momentum $d\vec{L}_z = \vec{r} \times (dm\vec{v}) = r^2 dm\vec{\omega}$. It is along the z-axis i.e. axis of rotation. In the next figure total angular momentum $\vec{L}_z = \int d\vec{L}_z = I_z\vec{\omega}$ about the axis of rotation is shown. It is also along the axis of rotation.





Relation between torque & angular momentum

For a rotating body

$$\vec{L} = I\vec{\omega} \Rightarrow \quad \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} \quad [\because \text{ I is constant.}]$$

$$\frac{d\vec{L}}{dt} = I\vec{\alpha} = \vec{\tau} \quad \left[\because \frac{d\vec{\omega}}{dt} = \vec{\alpha} \text{ and } \vec{\tau}=I\vec{\alpha}\right]$$

The rate of change of angular momentum is equal to the net torque acting on the body. This expression is the rotational analogue of $\frac{d\vec{p}}{dt} = \vec{F}$ and hence is referred to as newtons II law for rotational motion.

Angular Impulse & Angular Impulse–Momentum principle

If a large torque acts on a body for a small time then angular impulse = $\int \vec{\tau} dt$

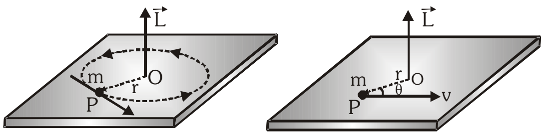
Angular impulse $= \vec{\tau}_{av}\Delta t = \Delta\vec{L}$ = change in angular momentum $\left[\because \vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}\right]$

Action of angular impulse is to change the angular momentum. It has same units, dimensions and direction as the angular momentum.

So angular impulse given to a rigid body is equal to the change in its angular momentum. This statement is also known as angular impulse-momentum principle.

GOLDEN KEY POINTS

- Angular momentum is an axial vector.



- As torque $(\vec{r} \times \vec{F})$ is defined as the 'moment of force', Angular momentum $(\vec{r} \times \vec{p})$ is also define as the moment of linear momentum.
- In cartesean coordinates, angular momentum is expressed as :

$$\vec{L} = (\vec{r} \times \vec{p}) = m(\vec{r} \times \vec{v}) \; (\because \vec{p} = m\vec{v}) \;\; = m\left[(x\hat{i} + y\hat{j} + z\hat{k}) \times (v_x\hat{i} + v_y\hat{j} + v_z\hat{k})\right]$$

$$\vec{L} = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

i.e. $\vec{L}=m[\,\hat{i}\,(yv_z-zv_y) - \hat{j}\,(xv_z-zv_x) + \hat{k}\,(xv_y-yv_x)]$

- As the magnitude of the angular momentum is $L = mvr\sin\theta$

  (a) For $\theta = 0°$ or $180°$ i.e. $\vec{r}$ and $\vec{v}$ are parallel or anti–parallel $\therefore \sin\theta = 0$; L will be minimum.

  If the axis of rotation is on the line of motion the of moving particle then angular momentum is minimum and zero.

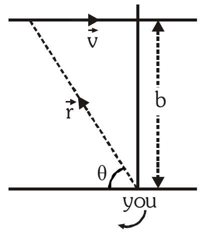
  (b) For $\theta = 90°$, i.e., angular momentum is maximum when $\vec{r}$ and $\vec{v}$ are orthogonal $\therefore \sin\theta = 1$; L will be maximum.

  i.e. in case of circular motion of a particle, angular momentum is maximum and is mvr.

- As $|\vec{L}| = mvr\sin\theta$ so if the point about which angular momentum is to be determined is not on the line of motion i.e., $\theta \neq 0$ or $180°$ then $|\vec{L}| \neq 0$ i.e., a particle in translatory motion always has an angular momentum unless the point is on the line of motion

- Direction of angular momentum due to linear motion :

  Imagine your position at the point about which you want to determine the angular momentum spread the palm of your right hand along the position vector and curl the fingers in the direction of velocity of particle then the thumb of your right hand shows the direction of angular momentum (Right hand thumb rule).



  $\vec{L} = mvb(-\hat{k})$ here $b = r\sin\theta$

  $\vec{L} = mvr\sin\theta(-\hat{k}) \Rightarrow \vec{L} = m(\vec{r}\times\vec{v})$

  a particle is moving parallel to X–axis or Y–axis or along any straight line with constant velocity then its angular momentum about the origin will remain constant.

Illustrations

Illustration 35.

A particle of mass 0.01 kg having position vector $\vec{r} = (10\hat{i} + 6\hat{j})$ metre is moving with a velocity $5\hat{i}$ m/s. Calculate its angular momentum about the origin.

Solution

$\vec{p} = m\vec{v} = 0.01\times 5\hat{i} = 0.05\hat{i} \Rightarrow \vec{L} = \vec{r}\times\vec{p} = (10\hat{i} + 6\hat{j})\times 0.05\hat{i} = 0.3(-\hat{k}) = -0.3\hat{k}$ J/s.

Illustration 36.

A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad/s; the radius of the cylinder is 0.25 m. What is the magnitude of the angular momentum of the cylinder about its axis ?

Solution

M = 20 kg, ω = 100 rad/s R = 0.25 m

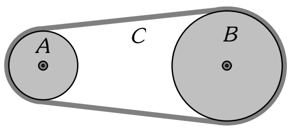
Moment of inertia of the cylinder about its axis

$I = \frac{1}{2}MR^2 = \frac{1}{2}\times 20\times(0.25)^2 = 0.625$ kg-m^2

Angular momentum $L = I\omega = 0.625\times 100 = 62.5$ J-s

Illustration 37.

A belt moves over two pulleys A and B as shown in the figure. The pulleys are mounted on two fixed horizontal axles. Radii of the pulleys A and B are 50 cm and 80 cm respectively. Pulley A is driven at constant angular acceleration of 0.8 rad/s^2 until pulley B acquires an angular velocity of 10 rad/s. The belt does not slide on either of the pulleys.



(a) Find the acceleration of a point C on the belt and angular acceleration of the pulley B.

(b) How long does it take for the pulley B to acquire an angular velocity of 10 rad/s ?

Solution.

Since the belt does not slide on the pulleys, magnitudes of velocity and acceleration of any point on the belt are same as that of any point on the periphery of either of the two pulleys.

$\vec{a}_T = \vec{\alpha}\times\vec{r}$ $a_C = \alpha_A r_A = \alpha_B r_B$

Substituting r_A = 0.5 m, r_B = 0.8m and α_A = 0.8 rad/s^2,

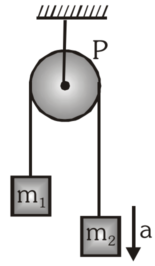
we have a_C = 0.4 m/s^2 and $\alpha_B = \frac{a_C}{r_B} = \frac{\alpha_A r_A}{r_B} = 0.5$ rad/s^2

$\because \quad \omega = \omega_o + \alpha t \Rightarrow t = \frac{\omega_B - \omega_{B_o}}{\alpha_B}$

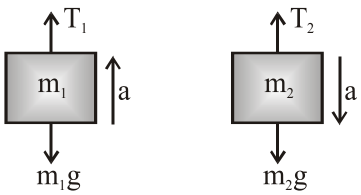
Substituting $\omega_{Bo} = 0$, $\omega_B = 10$ rad/s and $\alpha_B = 0.5$ rad/s^2, we have t = 20 s

Illustration 38.

In the figure, the blocks have unequal masses m_1 and m_2 ($m_1 < m_2$). m_2 has a downward acceleration a. The pulley P has a radius r, and some mass. The string does not slip on the pulley. Find the acceleration of block and angular acceleration of pulley. (I = moment of inertia of pulley)



Solution.



$T_1 - m_1g = m_1a$ ....(1)

$m_2g - T_2 = m_2a$ ....(2)

Total torque $= I\alpha$

$(T_2 - T_1)R = I\alpha$ ....(3)

$a = \alpha R$ ....(4)

By Eq. (1), (2) (3) and (4) $\Rightarrow a = \dfrac{(m_2 - m_1)}{m_1 + m_2 + \dfrac{I}{r^2}} g \Rightarrow \alpha = \dfrac{a}{R}$

Illustration 39.

The diameter of a disc is 0.5 m and its mass is 16 kg. What torque will increase its angular velocity from zero to 120 rotations/minute in 8 seconds ?

Solution.

Moment of inertia of the disc $I = \dfrac{1}{2}MR^2 = \dfrac{1}{2} \times 16 \times \left[\dfrac{0.5}{2}\right]^2 = \dfrac{1}{2}$ kg-m²

Angular velocity $\omega = \dfrac{120 \times 2\pi}{60} = 4\pi$ rad/s and change in angular momentum $\Delta L = I\omega - 0 = I\omega$

$\therefore$ Angular impulse is $\tau \times t = \Delta L \Rightarrow \tau = \dfrac{\Delta L}{t} = \dfrac{1}{8} \times \dfrac{1}{2} \times 4\pi = \dfrac{\pi}{4}$ N–m

Illustration 40.

During the launch from a board, a diver's angular speed about her centre of mass changes from zero to 6.20 rad/s in 220 ms. Her rotational inertia about the centre of mass is 12.0 kg-m². During the launch, what is the magnitude of :

(a) her average angular acceleration

(b) the average external torque acting on her from the board?

Solution.

(a) From the kinematic equation $\omega = \omega_0 + \alpha t$, we get $\alpha = \dfrac{\omega - \omega_0}{t} = \dfrac{6.20 \text{ rad/s}}{220 \times 10^{-3}\text{s}} = 28.2 \text{ rad/s}^2$

(b) If I is the rotational inertia of the diver, then the magnitude of the torque acting on her is

$\tau = I\alpha = (12.0 \text{ kg.m}^2)(28.2 \text{ rad/s}^2) = 3.38 \times 10^2$ N-m.

Illustration 41.

A spherical shell has a radius of 1.90 m. An applied torque of 960 N-m gives the shell an angular acceleration of 6.20 rad/s^2 about an axis passing through the centre of the shell. What are

(a) the rotational inertia of the shell about that axis and (b) the mass of the shell?

Solution.

(a) Use $\tau = I\alpha$ where τ is the net torque acting on the shell, I is the rotational inertia of the shell,

and α as its angular acceleration. This gives $I = \frac{\tau}{\alpha} = \frac{960\ \text{N}\cdot\text{m}}{6.20\ \text{rad/s}^2} = 155$ kg-m^2

(b) The rotational inertia of the shell is given by $I = \left(\frac{2}{3}\right)MR^2 \therefore M = \frac{3I}{2R^2} = \frac{3\left(155\text{kg}\cdot\text{m}^2\right)}{2\left(1.90\ \text{m}\right)^2} = 64.4$ kg.

BEGINNER'S BOX-4

1. If the earth is a point mass of 6 × 10^{24} kg revolving around the sun at a distance of 1.5 × 10^8 km and in time of T= 3.14 × 10^7 seconds, then the angular momentum of the earth around the sun is :–

(A) 1.2 × 10^{18} kg-m^2/s (B) 1.8 × 10^{20} kg-m^2/s (C) 1.5 × 10^{37} kg-m^2/s (D) 2.7 × 10^{40} kg-m^2/s

2. If a particle moves along a straight line whose equation is y = mx then what will be its angular momentum about the origin ? Explain your answer.

3. Two gear wheels which are meshed together have radii of 0.50 cm and 0.15 cm. The number of revolutions undergone by the smaller one when the larger goes through 3 revolutions is :-

(A) 5 revolutions (B) 20 revolutions (C) 1 revolution (D) 10 revolutions

4. A uniform circular disc of mass 200 g and radius 4·0 cm is rotated about one of its diameter at an angular speed of 10 rad/s. Find the kinetic energy of the disc and its angular momentum about the axis of rotation.

12. CONSERVATION OF ANGULAR MOMENTUM (COAM)

If the angular impulse of all the external forces about an axis vanishes over a certain time interval, then the angular momentum of the system about the same axis remains unchanged in that time interval.

If $\Sigma\int_{t_1}^{t_2}\vec{\tau}_o dt = 0$, we have $\vec{L}_i = \vec{L}_f$

$$\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t} \Rightarrow \tau = \frac{L_f - L_i}{\Delta t} = \frac{I\omega_f - I\omega_i}{\Delta t} = \frac{I(\omega_f - \omega_i)}{\Delta t}$$

If the resultant external torque acting on a system is zero then the total angular momentum of the system remains constant.

If $\tau = 0$ then $\frac{\Delta L}{\Delta t} = 0 \Rightarrow$ L = constant $\Rightarrow L_f = L_i$ or $I_1\omega_1 = I_2\omega_2$

If a system is isolated from its surrounding any internal interaction between its different parts cannot alter its total angular momentum.

The condition of zero net angular impulse required for conservation of angular momentum can be realised in the following cases :

- If no external force acts, the angular impulse about all axes will be zero and hence angular momentum remains conserved about all axes.
- If net torque of all the external forces or torques of each individual force about an axis vanishes the angular momentum about that axis will be conserved.
- If all the external forces are finite in magnitude and the concerned time interval is infinitely small, the angular momentum remains conserved.

The principle of conservation of angular momentum governs a wide range of physical processes from subatomic to celestial world. The following examples explicate some of these applications.

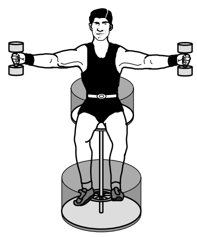
Examples Based On COAM

Spinning Ice Skater.

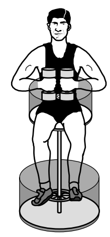
A spinning ice skater or a ballet dances can control her moment of inertia by extending or folding her arms and make use of conservation of angular momentum to perform their spins. In doing so no external force is needed and if we ignore effects of friction from the ground and the air, the angular momentum can be assumed to be conserved. When she spreads her hands or legs away from the spin axis, her moment of inertia increases therefore her angular velocity decreases and when she brings her hands or legs closer her moment of inertia decreases and consequently her angular velocity increases.

Student on rotating turntable

The student with the dumbbells and the turntable make an isolated system on which no external torque acts (if we ignore friction in the bearings of the turntable and air resistance). Initially, the student has his arms stretched. When he draws the dumbbells close to his body, his angular velocity increases due to conservation of angular momentum.

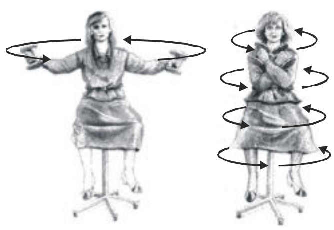


Larger moment of inertia and smaller angular velocity



Smaller moment of inertia and larger angular velocity

- If a lady skating on ice while spinning folds her arms then her M.I. decreases and consequently 'ω' increases.

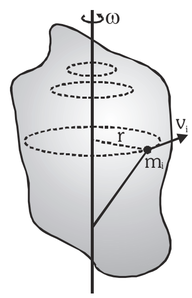


13. ROTATIONAL KINETIC ENERGY (KINETIC ENERGY OF ROTATION)

- The energy possessed due to rotational motion of a body is known as rotational kinetic energy.
- A rigid body is rotating about an axis with a uniform angular velocity ω. The body is assumed to be composed of particles of masses m_1, m_2 .....

The linear velocity of the particles are $v_1 = \omega r_1$, $v_2 = \omega r_2$, ...

Therefore the total kinetic energy of the rotating body is

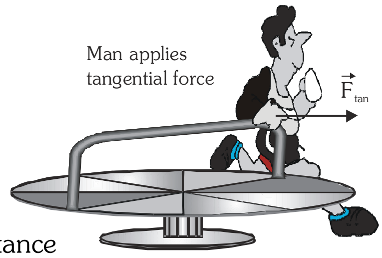


$$KE_r = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \ldots = \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + \ldots)\omega^2 = \frac{1}{2}I\omega^2 .$$

- It is a scalar quantity.

- Rotational kinetic energy $KE_r = \frac{1}{2}I\omega^2 = \frac{1}{2}I \times \frac{4\pi^2}{T^2} = \frac{1}{2}MK^2\omega^2 = \frac{1}{2}I \times \frac{v^2}{R^2} = \frac{L.\omega}{2} = \frac{L^2}{2I}$
- If external torque acting on a body is equal to zero ($\tau = 0$), L= constant $\Rightarrow E \propto \frac{1}{I} \propto \frac{1}{MK^2}$

Work and Power in Rotational Motion



Suppose a tangential force $\vec{F}_{tan}$ acts on the rim of a pivoted disc [for example, a man rotating a merry-go-round on a play ground, while simultaneously running along with it]. The disc rotates through an infinitesimal angle $d\theta$ about a fixed axis during an infinitesimal time interval dt.

The work done by the force $\vec{F}_{tan}$ while any point on the rim moves a distance ds is $dW = F_{tan}\ ds$.

If $d\theta$ the corresponding is angular displacement, then $ds = Rd\theta$ and $\therefore\ dW = F_{tan}Rd\theta$

Torque due to the force $\vec{F}_{tan}$ is $\tau = F_{tan}R$ $\qquad \therefore\ dW = \tau d\theta$ ....(i)

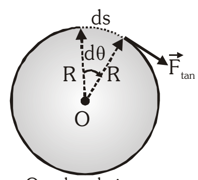
Total work W done by the torque during an angular displacement from θ_1 to θ_2 is :

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta \text{ (work done by a torque)} \qquad \text{....(ii)}$$

If the torque remains constant while the angle changes by a finite amount $\Delta\theta = \theta_2 - \theta_1$ then

$$W = \tau\ (\theta_2 - \theta_1) = \tau\ \Delta\theta \qquad \text{(work done by a constant torque)} \qquad \text{....(iii)}$$

Work done by a constant torque is the product of torque and the angular displacement.



Overhead view of merry-go-round

Let τ represent the net torque on the body so from equation $\tau = I\alpha$.

assuming the body to be rigid so that its moment of inertia I remains constant,

$$\tau d\theta = (I\alpha)d\theta = I\frac{d\omega}{dt}d\theta = I\frac{d\theta}{dt}d\omega = I\omega d\omega$$

Since τ is the net torque, the integral in equation (ii) yields the total work done on the rotating body.

$$W_{total} = \int_{\omega_1}^{\omega_2} I\omega\ d\omega = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

When a torque executes work on a rotating rigid body, the kinetic energy of the body changes by an amount equal to the work done.

- **Work-energy theorem in rotational motion**

Work done by the torque = Change in kinetic energy of rotation $W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$

The change in the rotational kinetic energy of a rigid body equals the work done by external torques exerted on the body. This equation is analogous to equation corresponding to the work–energy theorem applicable for a particle.

- **Rotational power**

The power associated with the work done by torque acting on a rotating body :

Divide both sides of equation $dW = \tau d\theta$ by the time interval dt during which the angular displacement $d\theta$ occurs.

$\therefore \quad \frac{dW}{dt} = \tau\frac{d\theta}{dt}$. Instantaneous rotational power $P_r = \tau\omega$. In general, $P_r = \vec{\tau}.\vec{\omega}$.

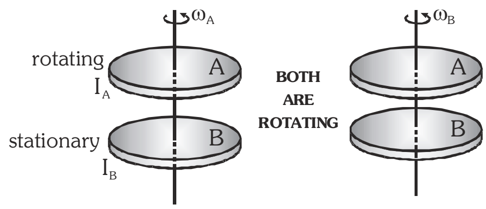
GOLDEN KEY POINTS

- If a system is isolated from its surrounding
  i.e. any internal interaction between different parts of a system cannot alter its total angular momentum.
  If $\tau = 0$ then $I_1\omega_1 = I_2\omega_2$
  $MK_1^2\ 2\pi n_1 = MK_2^2\ 2\pi n_2 \qquad (\because\ I = MK^2,\ \omega = 2\pi n)$
  $\Rightarrow \quad K_1^2\ n_1 = K_2^2\ n_2.$
- The angular velocity of a planet about the sun increases due to decrease in I when it comes closer to the sun.
- The speed of the inner layers of the whirlwind in a tornado is alarmingly high.
- If external torque on a system is zero, then its angular momentum remains conserved. However the rotational kinetic energy is not conserved.
  $I_1\omega_1 = I_2\omega_2 \quad \Rightarrow \frac{1}{2}I_1^2\omega_1^2 = \frac{1}{2}I_2^2\omega_2^2 \Rightarrow\ I_1 \times \frac{1}{2}I_1\omega_1^2 = I_2 \times \frac{1}{2}I_2\omega_2^2 \Rightarrow I_1K_{r_1} = I_2K_{r_2}$ If $I_1 > I_2$ then $K_{r_1} < K_{r_2}$ and vice-versa.
  So if the moment of inertia decreases, the rotational kinetic energy increases in absence of external torque and vice versa.
- A body rotating about a fixed axis essentially possesses rotating kinetic energy.
- Moment of inertia of a rigid body about a given axis is numerically equal to twice its rotational kinetic energy when it is rotating with unit angular velocity. $\left(I = \frac{2K_r}{\omega^2}\right)$
- Rotational kinetic energy depends upon the axis of rotation.

Illustrations

Illustration 42.

Two wheels A and B are placed coaxially. Wheel A has moment of inertia I_A and angular velocity ω_A while B is stationary. On clubbing them with a clutch they move jointly by an angular velocity 'ω' then find the M.I. of 'B'.



Solution.

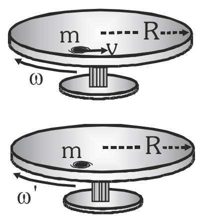
The phenomenon occurs in the absense of external torque, so applying law of COAM, we have

$$\because\ I_A\ \omega_A = (I_A + I_B)\ \omega \qquad \Rightarrow I_B = \frac{I_A\omega_A - I_A\omega}{\omega}$$

$$= I_A\left(\frac{\omega_A}{\omega} - 1\right)$$

Illustration 43.

A cockroach of mass 'm' is moving with velocity v in anticlockwise sense on the rim of a disc of radius R. The M.I. of the disc about the axis is 'I' and it is rotating in clockwise direction with an angular velocity 'ω'. If the cockroach stops then calculate the angular velocity of disc.



Solution.

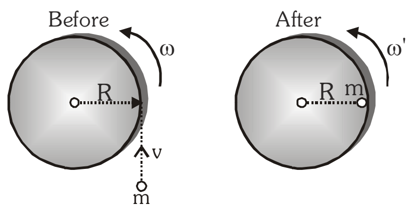
The phenomenon occurs in the absense of external torque, so applying law of COAM, we have

$$I_{disc}\ \omega - mvR = (I_{disc} + mR^2)\ \omega' \Rightarrow\ \omega' = \frac{I\omega - mvR}{I + mR^2}.$$

Illustration 44.

A solid cylinder of mass 'M' and radius 'R' is rotating along its axis with angular velocity ω without friction. A particle of mass 'm' moving with velocity v collides against the cylinder and sticks to its rim. After the impact calculate angular velocity of cylinder.

Solution.



Initial angular momentum of cylinder = $I\omega$

Initial angular momentum of particle = mvR

Before collision the total angular momentum $L_1 = I\omega + mvR$

After collision the total angular momentum $L_2 = (I + mR^2)\omega'$

$L_1 = L_2 \Rightarrow (I+mR^2)\omega' = I\omega+mvR.$

New angular velocity $\omega' = \dfrac{I\omega + mvR}{I+mR^2}$.

Note : Initial kinetic energy of the system $= \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$. Final kinetic energy of the system $= \frac{1}{2}(I+mR^2)\omega'^2$

Illustration 45.

Keeping the mass of earth constant if its radius is halved then calculate the duration of the day.

Solution.

$$I_1\omega_1 = I_2\omega_2 \Rightarrow \frac{2}{5}MR^2 \times \frac{2\pi}{T_1} = \frac{2}{5}M\left(\frac{R}{2}\right)^2 \times \frac{2\pi}{T_2}$$

$$\Rightarrow T_2 = \frac{T_1}{4} \qquad \because T_1 = 24 \text{ h} \qquad \therefore T_2 = 6 \text{ h}.$$

Illustration 46.

Explain with reason that if ice melts at polar region then moment of inertia of earth increases, angular velocity ω decreases and the duration of the day becomes longer.

Solution.

If ice in the polar region melts, the resulting water will flow towards the equator and consequently moment of inertia of earth will increase because more mass has accumulated at equator which is at more distant from the rotational axis as compared to the poles.

So from law of COAM if I increases then ω decreases. Time period (T) $= \dfrac{2\pi}{\omega} \Rightarrow$ T increases.

Due to increase in time period, the duration of day and night will increase.

Illustration 47.

A rotating table has angular velocity 'ω' and moment of inertia I_1. A person of mass 'm' stands on the centre of rotating table. If the person moves a distance r along its radius then what will be the final angular velocity of the rotating table ?

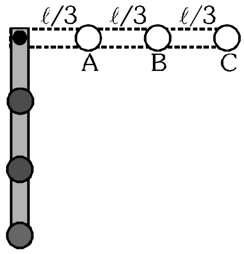
Solution.

The process takes place in the absense of external torque, so law of COAM can be applied,

Initial angular momentum = Final angular momentum $I_1\omega_1 = (I_1+mr^2)\,\omega_2 \Rightarrow \omega_2 = \dfrac{I_1\omega}{I_1+mr^2}$

Illustration 48.

A light rod carries three equal masses A, B and C as shown in the figure. What will be the velocity of B in the vertical position of the rod, if it is released from horizontal position as shown in the figure ?



(A) $\sqrt{\frac{8g\ell}{7}}$ (B) $\sqrt{\frac{4g\ell}{7}}$ (C) $\sqrt{\frac{2g\ell}{7}}$ (D) $\sqrt{\frac{10g\ell}{7}}$

Solution.

Applying law of conservation of mechanical energy

Loss in gravitational P.E. = Gain in rotational K.E. i.e.,

$$mg\frac{\ell}{3} + mg\left(\frac{2\ell}{3}\right) + mg\ell = \frac{1}{2}\left\{m\left(\frac{\ell}{3}\right)^2 + m\left(\frac{2\ell}{3}\right)^2 + m\ell^2\right\}\omega^2$$

$$\Rightarrow \omega = \sqrt{\frac{36g}{14\ell}} \Rightarrow v_B = \omega\ell_B = \frac{2\ell}{3}\sqrt{\frac{36g}{14\ell}} = \sqrt{\frac{8g\ell}{7}}.$$

Illustration 49.

A fly wheel (in shape of ring) of mass 0.2 kg and radius 10 cm is rotating with $\frac{5}{\pi}$ rev/s about an axis perpendicular to its plane passing through its centre. Calculate the angular momentum and kinetic energy of the fly wheel. **[AIPMT 2006]**

Solution.

Angular velocity $\omega = 2\pi \times \frac{5}{\pi} = 10$ rad/s

Moment of inertia $I = mr^2 = (0.2)\,(0.1)^2 = 2\times 10^{-3}$ kg-m^2

Angular momentum= $I\omega = 2 \times 10^{-3} \times 10 = 2\times 10^{-2}$ J-s or 2×10^{-2} kg-m^2/s

Kinetic energy $= \frac{1}{2}I\omega^2 = \frac{1}{2}\times 2\times 10^{-3}\times (10)^2 = 0.1$ J.

Illustration 50.

A point mass is tied to one end of a cord whose other end passes through a vertical hollow tube, caught in one hand. The point mass is being rotated in a horizontal circle of radius 2 m with a speed of 4 m/s. The cord is then pulled down so that the radius of the circle reduces to 1m. Compute the new linear and angular velocities of the point mass and also the ratio of kinetic energies in the initial and final states.

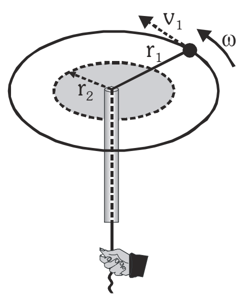
Solution.

The force on the point mass due to cord is radial and hence the torque about the centre of rotation is zero. Therefore, the angular momentum must remain constant as the cord is shortened.

Let mass of the particle be m let it rotate initially in circle of radius r_1 with linear velocity v_1 and angular velocity ω_1. Further let the corresponding quantities in the final state be radius r_2, linear velocity v_2 and angular velocity ω_2.

$\because$ Initial angular momentum = Final angular momentum

$$\therefore \quad I_1\omega_1 = I_2\omega_2 \Rightarrow mr_1^2\frac{v_1}{r_1} = mr_2^2\frac{v_2}{r_2} \Rightarrow r_1v_1 = r_2v_2$$

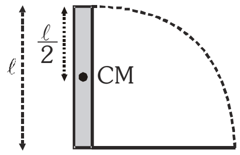


$$\therefore \quad v_2 = \frac{r_1}{r_2}v_1 = \frac{2}{1}\times 4 = 8 \text{ m/s} \quad \text{and} \quad \omega_2 = \frac{v_2}{r_2} = \frac{8}{1} = 8 \text{ rad/s.}$$

$$\frac{\text{Final K.E.}}{\text{Initial K.E.}} = \frac{\frac{1}{2}I_2\omega_2^2}{\frac{1}{2}I_1\omega_1^2} = \frac{mr_2^2\times\left[\frac{v_2}{r_2}\right]^2}{mr_1^2\times\left[\frac{v_1}{r_1}\right]^2} = \frac{v_2^2}{v_1^2} = \frac{(8)^2}{(4)^2} = 4.$$

Illustration 51.

A thin meter scale is kept vertical by placing its lower end hinged on floor. It is allowed to fall. Calculate the velocity of its upper end when it hits the floor .



Solution.

Loss in PE $\left(\frac{mg\ell}{2}\right)$ = gain in rotational $KE\left(\frac{1}{2}I\omega^2 = \frac{1}{2}\frac{m\ell^2}{3} \times \frac{v^2}{\ell^2}\right) \Rightarrow v = \sqrt{3g\ell}$.

Illustration 52.

If the rotational kinetic energy of a body is increased by 300% then determine the percentage increase in its angular momentum.

Solution.

Percentage increase in angular momentum = $\frac{L_2 - L_1}{L_1} \times 100$

Now, $L = \sqrt{2IE}$ $\quad\therefore\ L \propto \sqrt{E} \Rightarrow\ E_1 = E$ and $E_2 = E + \frac{300}{100}E \Rightarrow E_2 = 4E$

Increase in angular momentum = $\frac{\sqrt{E_2} - \sqrt{E_1}}{\sqrt{E_1}} \times 100 = \frac{\sqrt{4E} - \sqrt{E}}{\sqrt{E}} \times 100 = 100\%$.

Illustration 53.

The power output of an automobile engine is advertised to be 200 H.P. at 6000 rpm. What is the corresponding torque generated.

Solution.

$$P = 200\ \text{H.P.} = 200 \times 746 = 1.49 \times 10^5\ \text{W}$$

& $\quad \omega = 6000$ rev/min $= 6000 \times \frac{2\pi}{60} = 628$ rad/s

$$\tau = \frac{P}{\omega} = \frac{1.49 \times 10^5}{628} = 237.5\ \text{N} - \text{m}$$

BEGINNER'S BOX-5

1. The angular velocity of a body changes from ω_1 to ω_2 without applying external torque. Then find the ratio of initial radius of gyration to final radius of gyration.

2. A stone is attached to a string and is revolved in a circular path with constant angular velocity ω. In this state its angular momentum is L. If the length of the string is reduced to half and again it is revolved with the same angular velocity ω then find its angular momentum.

3. A thin uniform thin rod of mass m and length ℓ is suspended from one of its ends and is rotated at the rate of f rotations per second. Find the rotational kinetic energy of the rod.

4. A wheel of moment of inertia 10 kg-m² rotates at the rate of 10 revolutions per minute. Find the work done in increasing its speed to 5 times of its initial value.

5. A solid ball of mass 1 kg and radius 3 cm is rotating about its own axis with an angular velocity of 50 radians per second. Find its kinetic energy of rotation.

6. When sand is poured gradually on the edge of a rotating disc, what will be the change in its angular velocity?

7. The moment of inertia of a wheel about its axis of rotation is 3.0 MKS units. Its kinetic energy is 600 J. Find its period of rotation.

8. The moment of inertia of a wheel is 1000 kg–m^2. At a given instant, its angular velocity is 10 rad/s. After the wheel rotates through an angle of 100 radians, its angular velocity increases to 100 rad/s. Calculate the,

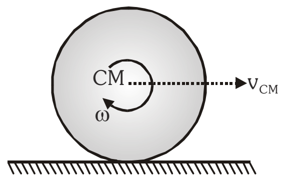
(a) torque applied on the wheel. (b) increase in the rotational kinetic energy.

14. ROLLING MOTION

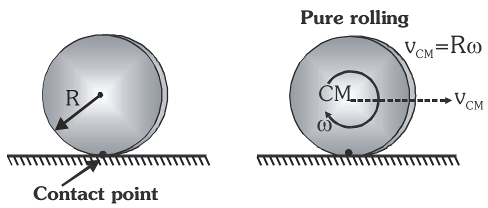
When a body performs translatory motion as well as rotatory motion combinedly then it is said to undergo rolling motion. The velocity of the centre of mass represents linear motion while angular velocity about the centroidal axis represents rotatory motion.

Total energy in rolling = Translatory kinetic energy + Rotatory kinetic energy.

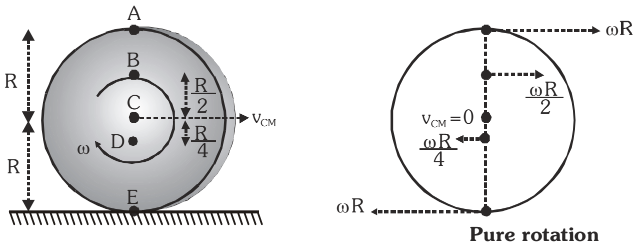
- **Rolling without slipping (Pure rolling)**



- If the relative velocity of the point of contact of the rolling body with the surface is zero then it is known as pure rolling.
- If a body is performing rolling then the velocity of any point of the body with respect to the surface is given by



$\vec{v} = \vec{v}_{CM} + \vec{\omega} \times \vec{R}$



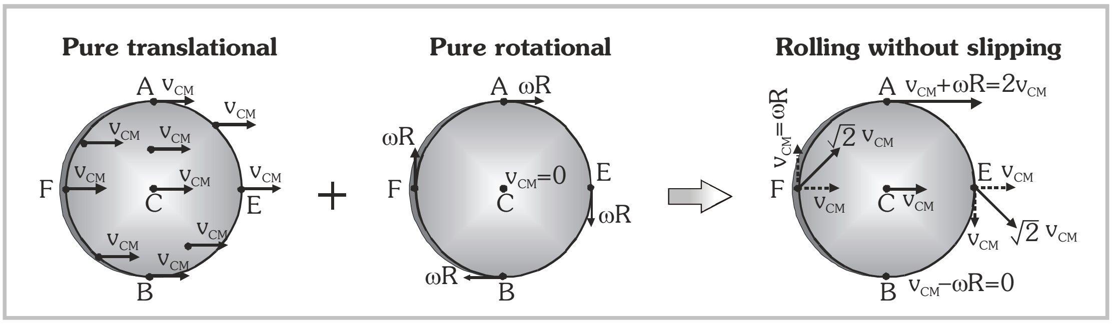
$$v_A = v_{CM} + \omega R = v_{CM} + v_{CM} = 2v_{CM}$$

$$v_B = v_{CM} + \frac{\omega R}{2} = v_{CM} + \frac{v_{CM}}{2} = \frac{3}{2}v_{CM}$$

$$v_C = v_{CM} + 0 = v_{CM}$$

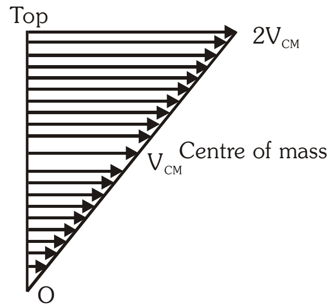
$$v_D = v_{CM} - \frac{\omega R}{4} = v_{CM} - \frac{v_{CM}}{4} = \frac{3}{4}v_{CM}$$

$$v_E = v_{CM} - \omega R = v_{CM} - v_{CM} = 0$$



Pure Translatory motion + Pure Rotatory Motion = Rolling motion

For pure rolling motion of the above mentioned body, we have



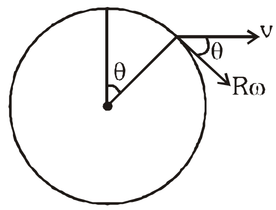
$$v_A = 2v_{CM}$$

$$v_E = \sqrt{2}\ v_{CM}$$

$$v_F = \sqrt{2}\ v_{CM}$$

$$v_B = 0$$

- **Velocity at a point on the rim of a rolling sphere**



$$v_{net} = \sqrt{v^2 + R^2\omega^2 + 2vR\omega\cos\theta}$$

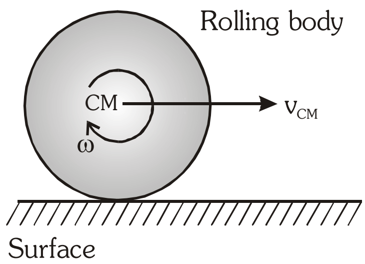
For pure rolling v = Rω

$$= \sqrt{v^2 + v^2 + 2v^2\cos\theta} \quad = \sqrt{2v^2 + 2v^2\cos\theta}$$

$$= \sqrt{2v^2(1+\cos\theta)} = 2v\cos\frac{\theta}{2} \quad \left(\text{since } 1 + \cos\theta = 2\cos^2\frac{\theta}{2}\right),$$

$$\boxed{v_{net} = 2v\cos\frac{\theta}{2}}$$

- **Rolling Kinetic Energy**



Rolling Kinetic Energy $E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \Rightarrow \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$

Rolling Kinetic Energy $E = \frac{1}{2}mv^2\left(1 + \frac{K^2}{R^2}\right)$

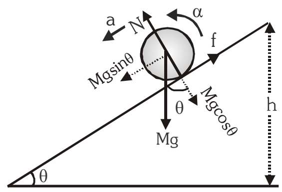
$$E_{translation} : E_{rotation} : E_{Total} = 1 : \frac{K^2}{R^2} : 1 + \frac{K^2}{R^2}$$

Body	$\frac{K^2}{R^2}$	$\frac{E_{trans}}{E_{rotation}} = \frac{1}{\left(\frac{K^2}{R^2}\right)}$	$\frac{E_{trans}}{E_{total}} = \frac{1}{\left(1+\frac{K^2}{R^2}\right)}$	$\frac{E_{rotation}}{E_{total}} = \frac{K^2/R^2}{\left(1+\frac{K^2}{R^2}\right)}$
Ring	1	1	$\frac{1}{2}$	$\frac{1}{2}$
Disc	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Solid sphere	$\frac{2}{5}$	$\frac{5}{2}$	$\frac{5}{7}$	$\frac{2}{7}$
Spherical shell	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Solid cylinder	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Hollow cylinder	1	1	$\frac{1}{2}$	$\frac{1}{2}$

Angular momentum of a body in combined translational and rotational motion

Suppose a body is rotating about an axis passing through its centre of mass with an angular velocity ω_{cm} and moving translationally with a linear velocity v_{CM}. Then, the angular momentum of the body about a point P outside the body in the lab frame is given by, $\vec{L}_P = \vec{L}_{cm} + \vec{r} \times \vec{p}_{cm}$, where $\vec{r}$ is the position vector of the centre of mass with respect to point P. Hence, $\vec{L}_P = \vec{L}_{cm} + \vec{r} \times m\vec{v}_{cm}$

15. ROLLING MOTION ON AN INCLINED PLANE



A body of mass M and radius R is rolling down a plane inclined at an angle θ with the horizontal. The body rolls without slipping. The centre of mass of the body moves in a straight line. External forces acting on the body are :

- Weight Mg of the body vertically downwards through its center of mass.
- The normal reaction N of the inclined plane.
- The frictional force f acting upwards and parallel to the inclined plane.

For linear motion $Mg\sin\theta - f = Ma_{CM}$; For angular motion $\tau = fR = I\alpha$

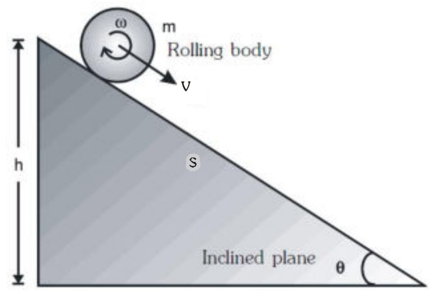
From the condition for pure rolling $a_{CM} = \alpha R \Rightarrow Mg\sin\theta - f = M\left(\frac{fR^2}{I}\right) = M\left(\frac{fR^2}{MK^2}\right) \Rightarrow f = \frac{Mg\sin\theta}{\left(1+\frac{R^2}{K^2}\right)}$

But $f \le \mu Mg\cos\theta \Rightarrow \frac{Mg\sin\theta}{\left(1+\frac{R^2}{K^2}\right)} \le \mu Mg\cos\theta \Rightarrow \mu \ge \frac{\tan\theta}{\left(1+\frac{R^2}{k^2}\right)} \Rightarrow$ for pure rolling $\mu_{min} = \frac{\tan\theta}{\left(1+\frac{R^2}{K^2}\right)}$

Rolling Motion on an inclined plane

(i) Velocity at the bottom of the inclined plane

Applying Conservation of mechanical energy principle



$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$$

$$mgh = \frac{1}{2}mv^2\left(1+\frac{K^2}{R^2}\right) \quad ...(1)$$

$$h = s\sin\theta \quad ...(2)$$

from (1) & (2)

$$\boxed{v_{Rolling} = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}} = \sqrt{\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}}}$$

When the body slides $\frac{K^2}{R^2} = 0$

∴ Velocity when the body slides $\boxed{v_{sliding} = \sqrt{2gh} = \sqrt{2gs\sin\theta}} \Rightarrow v_{sliding} > v_{rolling}$

(ii) Acceleration of the body

$v^2 = u^2 + 2as$ $(\because u = 0)$

∴ $v^2 = 2as$

$$\frac{2gs\sin\theta}{\left(1+\frac{K^2}{R^2}\right)} = 2as \Rightarrow \boxed{a_{rolling} = \frac{g\sin\theta}{\left(1+\frac{K^2}{R^2}\right)}}$$

When body slides $\frac{K^2}{R^2} = 0$

Acceleration when the body slides $a_{sliding} = g\sin\theta \Rightarrow a_{sliding} > a_{rolling}$

(iii) Time taken by the rolling body to reach the bottom

$s = \frac{1}{2}at^2 \quad \Rightarrow \quad t = \sqrt{\frac{2s}{a}}, \quad s = \frac{h}{\sin\theta}$

$$\boxed{t_{rolling} = \sqrt{\frac{2s}{g\sin\theta}\left(1+\frac{K^2}{R^2}\right)} = \sqrt{\frac{2h}{g\sin^2\theta}\left(1+\frac{K^2}{R^2}\right)} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}\left(1+\frac{K^2}{R^2}\right)}}$$

When the body slides, $\frac{K^2}{R^2} = 0$

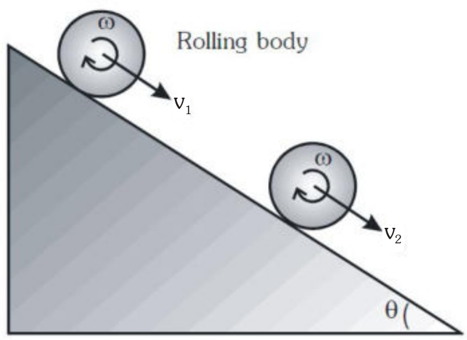
Time of descent when the body slides $\boxed{t_{sliding} = \sqrt{\frac{2s}{g\sin\theta}} = \sqrt{\frac{2h}{g\sin^2\theta}} = \frac{1}{\sin\theta}\sqrt{\frac{2h}{g}}}$

so $t_{rolling} > t_{sliding}$

Note If different bodies are allowed to roll down on an inclined plane then the body with

(i) least $\frac{K^2}{R^2}$ will reach first (ii) maximum $\frac{K^2}{R^2}$ will reach last

(iii) equal $\frac{K^2}{R^2}$ will reach together.



(iv) Change in kinetic energy due to rolling $v_2 > v_1$

$$= \frac{1}{2}mv_2^2\left(1+\frac{K^2}{R^2}\right) - \frac{1}{2}mv_1^2\left(1+\frac{K^2}{R^2}\right)$$

$$= \frac{1}{2}m\left(1+\frac{K^2}{R^2}\right)\left(v_2^2 - v_1^2\right)$$

(v) Rolling Vs Sliding

$\boxed{v_{rolling} = \frac{v_{sliding}}{\sqrt{1+\frac{K^2}{R^2}}}}$, $\boxed{a_{rolling} = \frac{a_{sliding}}{1+\frac{K^2}{R^2}}}$, $\boxed{t_{rolling} = t_{sliding}\sqrt{1+\frac{K^2}{R^2}}}$

$v_{rolling} < v_{sliding}$ $\quad a_{rolling} < a_{sliding}$ $\quad t_{rolling} > t_{sliding}$

Comparision between formula of translatory motion and rotatory motion

Translatory Motion	**Rotatory Motion**
• $\vec{F} = \frac{d\vec{p}}{dt}$	• $\vec{\tau} = \frac{d\vec{L}}{dt}$
• $\vec{F} = m\vec{a}$	• $\vec{\tau} = I\vec{\alpha}$
• Linear momentum $(\vec{p})$, $\vec{p} = m\vec{v}$	• Angular momentum $(\vec{L})$, $\vec{L} = I\vec{\omega}$
• Translational kinetic energy, $KE = \frac{1}{2}mv^2$	• Rotational kinetic energy, $E = \frac{1}{2}I\omega^2$
• Work done $W = \vec{F}.\vec{s}$ (constant force)	• Work done $W = \vec{\tau}.\vec{\theta}$ (constant torque)

- $W = \int \vec{F}.d\vec{s}$ Variable force
- Power

$$P = \frac{dW}{dt} = \frac{\vec{F}.d\vec{s}}{dt} = \vec{F}.\vec{v}$$

- Work–energy theorem

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

- Linear impulse

It is the product of large force and the small time for which it acts

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$$

Impulse–momentum theorem

$$\Delta\vec{p} = \vec{F}\Delta t = \text{Impulse}$$

- $W = \int \vec{\tau}.d\vec{\theta}$ Variable torque
- Power

$$P = \frac{dW}{dt} = \frac{\vec{\tau}.d\vec{\theta}}{dt} = \vec{\tau}.\vec{\omega}$$

- Work–energy theorem

$$W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$$

- Angular impulse

It is product of large torque for small time for which it acts

$$\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}$$

Angular Impulse–momentum theorem

$$\Delta\vec{L} = \vec{\tau}\Delta t = \text{Angular impulse}$$

GOLDEN KEY POINTS

- For pure rolling there may or may not be friction on surface.
- On a fixed smooth inclined surface pure rolling cannot sustain.
- The displacement of point of contact with the surface is equal to zero in pure rolling so work done is equal to zero.

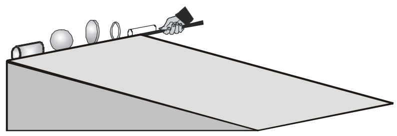
  Angular momentum is conserved about point of contact in pure rolling.

  For inclined plane :

  (i) Velocity of falling and sliding bodies along an inclined plane are equal and is more than that of a rolling body.

  (ii) Acceleration is maximum in case of falling and minimum in case of rolling.

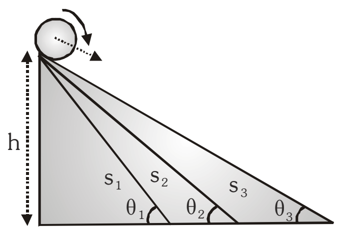
  (iii) Falling body reaches the bottom first while rolling body is the last to reach.
- If different bodies are allowed to roll down an inclined plane then body which has



→ least, $\frac{K^2}{R^2}$ will reach first

→ maximum, $\frac{K^2}{R^2}$ will reach last

→ equal, $\frac{K^2}{R^2}$ will reach together



- From the figure $\theta_3 < \theta_2 < \theta_1$ $a_3 < a_2 < a_1$

  $t_3 > t_2 > t_1$ $v_1 = v_2 = v_3$
- When a body performs pure rolling then its motion is pure rotatory about an axis passing through the point of contact, parallel to the surface and perpendicular to the direction of motion. It is known as the **instantaneous axis of rotation**.
- When a ring, disc, hollow sphere and a solid sphere roll on the same inclined plane then

  $v_S > v_D > v_H > v_R$

  $a_S > a_D > a_H > a_R$

  $t_S < t_D < t_H < t_R$

Illustration 54.

A thin hollow cylinder

(a) slides without rotating with a speed v. (b) rolls with the same speed without slipping.

Find the ratio of kinetic energies in the two cases.

Solution

For thin hollow cylinder $\frac{K^2}{R^2} = 1$ [analogous to a as ring]

(a) $E_{trans.} = \frac{1}{2}Mv^2$

(b) $E_{rolling} = \frac{1}{2}Mv^2\left[1+\frac{K^2}{R^2}\right] = \frac{1}{2}Mv^2\ (1+1) = Mv^2$

$$\frac{E_{trans.}}{E_{rolling}} = \frac{\frac{1}{2}Mv^2}{Mv^2} = \frac{1}{2} = 1 : 2.$$

Illustration 55.

When a sphere of moment of inertia 'I' rolls down an inclined plane then find the percentage of rotational kinetic energy of the total energy.

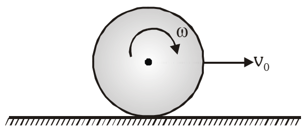
Solution

$$\frac{E_r}{E_t}\times 100 = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}Mv^2+\frac{1}{2}I\omega^2}\times 100 = \frac{\frac{2}{5}MR^2\times\frac{v^2}{R^2}}{Mv^2\left[1+\frac{K^2}{R^2}\right]}\times 100 = \frac{\frac{2}{5}}{1+\frac{2}{5}}\times 100 = 28.6\%$$

Illustration 56.

A solid sphere rolls without slipping on a rough surface and the centre of mass has a constant speed v_0. If the mass of the sphere is m and its radius is R, then find the angular momentum of the sphere about the point of contact.

Solution



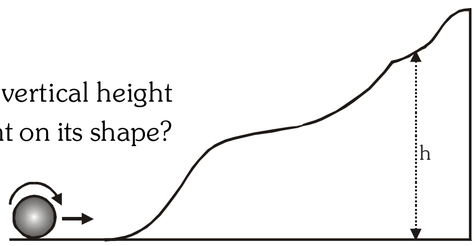
$\because\ \vec{L}_P = \vec{L}_{cm} + \vec{r}\times\vec{p}_{cm} = I_{cm}\vec{\omega} + \vec{R}\times m\vec{v}_{cm}$; here $v_{cm} = v_0$

Since sphere is in pure rolling motion hence $\omega = v_0/R$

$$\Rightarrow\ \vec{L}_p = \left(\frac{2}{5}MR^2\frac{v_0}{R}\right)\left(-\hat{k}\right) + Mv_0R\left(-\hat{k}\right) = \frac{7}{5}Mv_0R\left(-\hat{k}\right)$$

Illustration 57.

A body of mass M and radius r, rolling with velocity v on a smooth horizontal floor, rolls up a rough irregular inclined plane up to a vertical height $(3v^2/4g)$. Compute the moment of inertia of the body and comment on its shape?



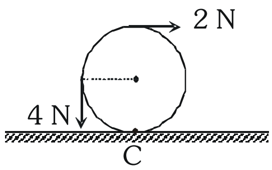
Solution

The total kinetic energy of the body $E = E_t + E_r = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \Rightarrow E = \frac{1}{2}Mv^2\ [1 + (I/Mr^2)]$ [as $v = r\omega$]

When it rolls up on an irregular inclined plane of height $h = (3v^2/4g)$, its KE is fully converted into PE, so by conservation of mechanical energy $\frac{1}{2}Mv^2\left[1+\frac{I}{Mr^2}\right] = Mg\left[\frac{3v^2}{4g}\right]$ which on simplification gives $I = (1/2)\ Mr^2$. This result clearly indicates that the body is either a disc or a cylinder.

Illustration 58.

A uniform solid disc of mass 1 kg and radius 1 m is kept on a rough horizontal surface. Two forces of magnitudes 2 N and 4 N have been applied on the disc as shown in the figure. If there is no slipping then the linear acceleration of the centre of mass of the disc is :-



(A) 4 m/s^2 (B) 2 m/s^2 (C) 1 m/s^2 (D) zero

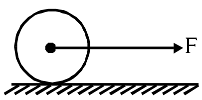
Solution

Taking torque about contact point C we have , $\tau = 4 \times R - 2 \times 2R = 0$, $F_{net} = 0$

Since $ma_{cm} = F_{net}$ $\therefore a_{cm} = 0$.

Illustration 59.

A horizontal force F acts on a sphere of mass M at its centre as shown. Coefficient of friction between the ground and the sphere is μ . What is maximum value of F, for which there is no slipping ?



Solution

For linear motion $F - f = Ma$

and for rotational motion $\tau = I\alpha$

$$\Rightarrow f.R = \frac{2}{5}MR^2 \cdot \frac{a}{R} \Rightarrow f = \frac{2}{5}Ma \text{ or } Ma = \frac{5}{2}f$$

$$\therefore F - f = \frac{5}{2}f \quad \text{or } f = \frac{2F}{7} \quad \because f \le \mu Mg \quad \therefore \frac{2F}{7} \le \mu Mg \quad \text{so } F \le \frac{7}{2}\mu Mg.$$

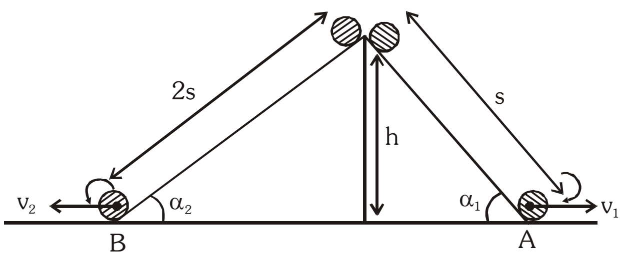
BEGINNER'S BOX-6

1. A ring, a disc and a solid sphere are simultaneously released to roll down from the top of a rough inclined plane of height h. Write the order in which these three bodies will reach the bottom ?

2. A hollow cylinder is rolling down an inclined plane which is inclined at an angle of 30° to the horizontal. Find its speed after travelling a distance of 10 m.

3. A wheel is rotating about a fixed axis. Find the moment of inertia of the wheel about the axis of rotation, when its angular speed is 30 radians/s and its kinetic energy is 360 joules.

4. Two uniform identical discs roll down on two inclined planes of length s and 2s respectively as shown in the figure. Find the ratio of velocities of the two discs at the points A and B of the inclined planes ?



5. Translational kinetic energy of a rolling hollow sphere is .... percent of its total energy.

ANSWERS

BEGINNER'S BOX-1

1. $\theta = 22$ rad., $\alpha = 2.0$ rad/s^2

2. $\omega = 22$ rad/s; $\theta = 24$ radians **3.** $\frac{A^2}{2B}$

4. (a) 120 rad/s; (b) (i) $-\frac{90}{\pi}$ rad/s^2 ; (ii) 24π m

5. (A) **6.** (A)

7. (B) **8.** (A)

BEGINNER'S BOX-2

1. 48 kg-m^2 **2.** $1:\sqrt{2}$ **3.** 1:4

4. 3 Kg-m^2 **5.** $\frac{6I}{5}$

6. Moment of inertia of a hollow cylinder will be larger as compare to a disc because most of the mass of the former is located away from the axis of rotation as compared to the latter.

7. Spoke do not carry much mass. Most of the mass is located at the rim. This gives cycle wheel more moment of inertia for same mass.

8. $I_B = 64I_A$ **9.** $\frac{\mu\ell^4}{6}$

10. $I = \frac{a^2}{4}(m_2 + m_3)$ **11.** $\frac{3}{2}MR^2$

12. It will be again $= \frac{2}{5}MR^2$, because the axis in question is also diameter of the sphere.

BEGINNER'S BOX-3

1. $\vec{\tau} = 11\hat{i} - 7\hat{j} - 5\hat{k}$

2. About point B moment of inertia is less than A so it is easier to rotate about point B

3. (B) **4.** (D) **5.** (B) **6.** (D)

7. (D) **8.** (A)

BEGINNER'S BOX-4

1. (D)

2. The angular momentum of particle about origin will be zero because given straight line passes through origin, so line of action of velocity (momentum) also passes through this point.

3. (D)

4. 4×10^{-3} J ; 8×10^{-4} J-s

BEGINNER'S BOX-5

1. $\sqrt{\frac{\omega_2}{\omega_1}}$ **2.** $\frac{L}{4}$ **3.** $\frac{2}{3}m\ell^2\pi^2 f^2$

4. 131.6 J **5.** 0.45 J

6. Angular velocity will decrease. **7.** $T = \frac{\pi}{10}$

8. (a) 4.95×10^4 N–m (b) 4.95×10^6 J

BEGINNER'S BOX-6

1. Solid sphere, disc and then ring

2. 7 m/s **3.** 0.8 kg–m^2

4. $\frac{v_1}{v_2} = 1$ **5.** 60

EXERCISE-I (Conceptual Questions)

KINEMETICS OF ROTATIONAL MOTION

1. Which of the following pairs do not match :-

(1) rotational power–Joule/sec

(2) torque-Newton meter

(3) angular displacement-radian

(4) angular acceleration - radian/sec

2. All the particles of a rigid rotating body move in a circular path when the axis of rotation:-

(1) passes through any point in the body

(2) is situated outside the body

(3) situated any where

(4) passes through the centre of mass

3. On account of the earth rotating about its axis :-

(1) the linear velocity of objects at equator is greater than at other places.

(2) the angular velocity of objects at equator is more than that of objects at poles.

(3) the linear velocity of objects at all places at the earth is equal, but angular velocity is different.

(4) at all places the angular velocity and linear velocity are uniform.

4. The quantity not involved directly in rotational motion of the body is

(1) moment of inertia

(2) torque

(3) angular velocity

(4) mass

MOMENT OF INERTIA

5. The moment of inertia of a body about a given axis of rotation depends upon :-

(1) the distribution of mass

(2) distance of particle of body from the axis of rotation

(3) shape of the body

(4) all of the above

6. A fly wheel is so constructed that almost whole of its mass is concentrated at its rim, because:-

(1) it increases the moment of inertia of the fly-wheel

(2) it decreases the moment of inertia of the fly-wheel

(3) it increases the speed of the fly-wheel

(4) it increases the power of the fly-wheel

7. The moment of inertia of a solid cylinder about its own axis is the same as its moment of inertia about an axis passing through its centre of gravity and perpendicular to its length. The relation between its length L and radius R is

(1) $L = \sqrt{2}R$ (2) $L = \sqrt{3}R$

(3) L = 3R (4) L = R

8. The wheels of moving vehicles are made hollow in the middle and thick at the periphery, because

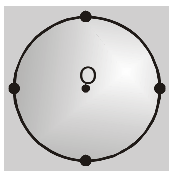
(1) it gives minimum moment of ineretia to the tyre

(2) its shape is a strong one

(3) this increases the speed

(4) it increases moment of inertia of tyre

9. Four similar point masses (each of mass m) are placed on the circumference of a disc of mass M and radius R. The M.I. of the system about the normal axis through the centre O will be:-



(1) $MR^2 + 4mR^2$ (2) $\frac{1}{2}MR^2 + 4mR^2$

(3) $MR^2 + \frac{8}{5}mR^2$ (4) none of these

10. By the theorem of perpendicular axes, if a body be in X-Z-plane then :-

(1) $I_x - I_y = I_z$ (2) $I_x + I_z = I_y$

(3) $I_x + I_y = I_z$ (4) $I_y + I_z = I_x$

11. The theorem of perpendicular axes is not applicable for determination of moment of inertia along the diameter, for which of the following body :-
(1) sphere (2) disc (3) ring (4) blade

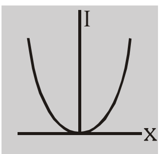
12. The axis X and Z in the plane of a disc are mutually perpendicular and Y-axis is perpendicular to the plane of the disc. If the moment of inertia of the body about X and Y axes is respectively 30 kg m^2 and 40 kg m^2 then M.I. about Z-axis in kg m^2 will be :-
(1) 70 (2) 50 (3) 10 (4) Zero

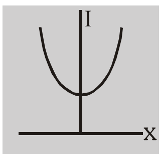
13. A solid sphere and a hollow sphere of the same mass have the same M.I. about their respective diameters. The ratio of their radii will be :-
(1) 1 : 2 (2) $\sqrt{3} : \sqrt{5}$
(3) $\sqrt{5} : \sqrt{3}$ (4) 5 : 4

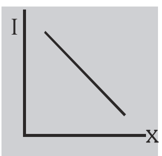
14. The moment of inertia of a square lamina about the perpendicular axis through its centre of mass is 20 kg-m^2. Then, its moment of inertia about an axis touching its side and in the plane of the lamina will be :-
(1) 10 kg-m^2 (2) 30 kg-m^2
(3) 40 kg-m^2 (4) 25 kg-m^2

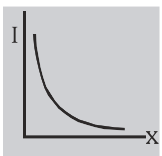
15. The M.I. of a thin rod of length ℓ about the perpendicular axis through its centre is I. The M.I. of the square structure made by four such rods about a perpendicular axis to the plane and through the centre will be :-
(1) 4I (2) 8I (3) 12I (4) 16I

16. The curve for the moment of inertia of a sphere of constant mass M versus distance of axis from its centre ?

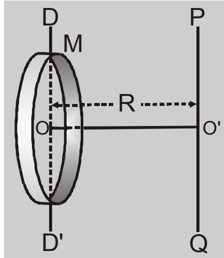
(1) 

(2) 

(3) 

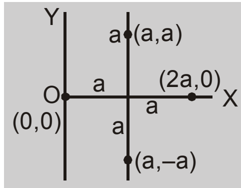
(4) 

17. The moment of inertia of a ring of mass M and radius R about PQ axis will be :-



(1) MR^2
(2) $\frac{MR^2}{2}$
(3) $\frac{3}{2}MR^2$
(4) $2MR^2$

18. Four point masses (each of mass m) are arranged it the X-Y plane the moment of inertia of this array of masses about Y-axis is



(1) ma^2
(2) $2ma^2$
(3) $4ma^2$
(4) $6ma^2$

19. Which of the following has the highest moment of inertia when each of them has the same mass and the same radius ?
(1) a hollow sphere about one of its diameters
(2) a solid sphere about one of its diameters
(3) a disc about its central axis perpendicular to the plane of the disc
(4) a ring about its central axis perpendicular to the plane of the ring.

20. If the radius of gyration of a solid disc of mass 10 kg about an axis is 0.40 m, then the moment of inertia of the disc about that axis is
(1) 1.6 kg m^2 (2) 3.2 kg m^2
(3) 6.4 kg m^2 (4) 9.5 kg m^2.

21. If I_1, I_2 and I_3 are moments of inertia of solid sphere, hollow sphere and a ring of same mass and radius about geometrical axis, which of the following statement holds good ?
(1) $I_1 > I_2 > I_3$ (2) $I_3 > I_2 > I_1$
(3) $I_2 > I_1 > I_3$ (4) $I_2 > I_3 > I_1$

22. Three thin uniform rods each of mass M and length L are placed along the three axis of a cartesian coordinate system with one end of each rod at the origin. The M. I. of the system about z-axis is
(1) $\frac{ML^2}{3}$ (2) $\frac{2ML^2}{3}$
(3) $\frac{ML^2}{6}$ (4) ML^2

23. Four particles each of mass m are placed at the corners of a square of side length ℓ.The radius of gyration of the system about an axis perpendicular to the square and passing through centre is :–

(1) $\frac{\ell}{\sqrt{2}}$ (2) $\frac{\ell}{2}$ (3) ℓ (4) $\ell\sqrt{2}$

24. The moment of inertia of a rod of mass M and length L about an axis passing through one edge and perpendicular to its length will be :–

(1) $\frac{ML^2}{12}$ (2) $\frac{ML^2}{6}$

(3) $\frac{ML^2}{3}$ (4) ML^2

25. A circular disc is to be made by using iron and aluminium so that it acquired maximum moment of inertia about geometrical axis. It is possible with:

(1) aluminium at interior and iron surrounded to it.

(2) iron at interior and aluminium surrounded to it.

(3) using iron and aluminium layers in alternate order.

(4) sheet of iron is used at both external surface and aluminium sheet as internal layer.

26. The moment of inertia in rotational motion is equivalent to :–

(1) angular velocity of linear motion

(2) mass of linear motion

(3) frequency of linear motion

(4) current

27. Two rods each of mass m and length ℓ are joined at the centre to form a cross. The moment of inertia of this cross about an axis passing through the common centre of the rods and perpendicular to the plane formed by them, is :–

(1) $\frac{m\ell^2}{12}$ (2) $\frac{m\ell^2}{6}$ (3) $\frac{m\ell^2}{3}$ (4) $\frac{m\ell^2}{2}$

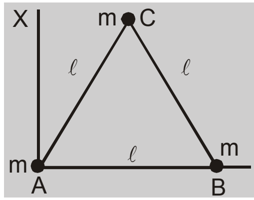
28. The ratio of the radii of gyration of a circular disc about a tangential axis in the plane of the disc and of a circular ring of the same radius about a tangential axis in the plane of the ring is :–

(1) 2 : 1 (2) $\sqrt{5} : \sqrt{6}$

(3) 2 : 3 (4) $1 : \sqrt{2}$

29. Three particles, each of mass m are situated at the vertices of an equilateral triangle ABC of side ℓ (as shown in the figure).

The moment of inertia of the system about a line AX perpendicular to AB and in the plane of ABC, will be :-



(1) $2\ m\ell^2$

(2) $\frac{5}{4}m\ell^2$

(3) $\frac{3}{2}m\ell^2$

(4) $\frac{3}{4}m\ell^2$

30. A solid cylinder of mass 20kg has length 1 m and radius 0.2 m. Then its moment of inertia (in kg-m^2) about its geometrical axis is :–

(1) 0.8 kg-m^2 (2) 0.4 kg-m^2

(3) 0.2 kg-m^2 (4) 20.2 kg-m^2

31. Moment of inertia :–

(1) is a vector quantity (2) is a scalar quantity

(3) is a tensor quantity (4) can not be calculate

32. The moment of inertia of a circular ring (radius R, mass M) about an axis which passes through tangentially and perpendicular to its plane will be:–

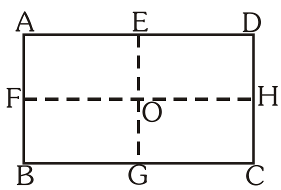
(1) $\frac{MR^2}{2}$ (2) MR^2 (3) $\frac{3}{2}MR^2$ (4) $2MR^2$

33. What is the moment of inertia of ring about its diameter ?

(1) MR^2 (2) $\frac{MR^2}{2}$

(3) $\frac{3}{4}MR^2$ (4) $\frac{5}{4}MR^2$

34. In the rectangular lamina shown in the figure, AB = BC/2. The moment of inertia of the lamina is minimum along the axis passing through :-



(1) AB

(2) BC

(3) EG

(4) FH

35. Which of the following bodies of same mass and same radius has minimum moment of inertia ?

(1) Ring

(2) Disc

(3) Hollow sphere

(4) Solid sphere

TORQUE, EQUILIBRIUM OF RIGID BODIES AND TOPPLING

36. A ring and a solid sphere of same mass and radius are rotating with the same angular velocity about their diameteric axes then :-

(1) it is easier to stop the ring

(2) it is easier to stop the solid sphere

(3) it is equally difficult to stop both of them

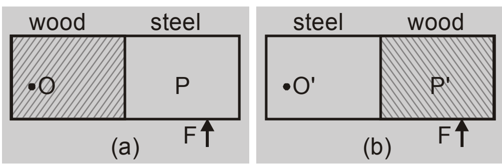
(4) it is not possible to stop a rotating body

37. For rotational motion, the Newton's second law of motion is indicated by :-

(1) $\omega_2^2 = \omega_1^2 + 2\alpha\theta$ (2) $\vec{F} = \frac{d\vec{p}}{dt}$

(3) $\vec{\tau} = \frac{d\vec{L}}{dt}$ (4) $\vec{F}_{12} = \vec{F}_{21}$

38. In the fig. (a) half of the meter scale is made of wood while the other half of steel. The wooden part is pivoted at O. A force F is applied at the end of steel part. In figure (b) the steel part is pivoted at O' and the same force is applied at the wooden end (In horizontal plane) :-



(1) more angular acceleration will be produced in (a)

(2) more angular acceleration will be produced in (b)

(3) same angular acceleration will be produced in both conditions

(4) information is incomplete

39. The moment of inertia of a disc of radius 0.5 m about its geometric axis is 2kg-m^2. If a string is tied to its circumference and a force of 10 Newton is applied, the value of torque with respect to this axis will be :-

(1) 2.5 N-m (2) 5 N-m

(3) 10 N-m (4) 20 N-m

40. In the above question, if the disc executes rotatory motion, its angular acceleration will be :-

(1) 2.5 rad/sec^2 (2) 5 rad/sec^2

(3) 10 rad/sec^2 (4) 20 rad/sec^2

41. In the above question, the value of its angular velocity after 2 seconds will be :-

(1) 2.5 rad/sec (2) 5 rad/sec

(3) 10 rad/sec (4) 20 rad/sec

42. In the above question, the change in angular momentum of disc in first 2 seconds (in Nm second) will be –

(1) 2.5 (2) 5 (3) 10 (4) 20

43. In the above question, angular displacement of the disc, in first two second will be (in radian) :-

(1) 2.5 (2) 5 (3) 10 (4) 20

44. A particle of mass m and radius of gyration K is rotating with an angular acceleration α. The torque acting on the particle is

(1) $\frac{1}{2}mK^2\alpha$ (2) $mK^2\alpha$

(3) mK^2/α (4) $\frac{1}{4}mK^2\alpha^2$

45. The grinding stone of a flour mill is rotating at 600 rad/sec. for this power of 1.2 k watt is used. the effective torque on stone in N–m will be :–

(1) 1 (2) 2 (3) 3 (4) 4

46. A rigid body is rotating about an axis. To stop the rotation, we have to apply :-

(1) pressure (2) force

(3) momentum (4) torque

47. A wheel has moment of inertia 5×10^{-3} kg m^2 and is making 20 rev/sec. The torque needed to stop it in 10 sec is .... $\times 10^{-2}$ N-m :–

(1) 2π (2) 2.5π (3) 4π (4) 4.5π

48. A wheel having moment of inertia 2 kg–m^2 about its vertical axis, rotates at the rate of 60 rpm about the axis. The torque which can stop the wheel's rotation in one minute would be :–

(1) $\frac{\pi}{12}$ N–m (2) $\frac{\pi}{15}$ N–m

(3) $\frac{\pi}{18}$ N–m (4) $\frac{2\pi}{15}$ N–m

49. A constant torque acting on a uniform circular wheel changes its angular momentum from A_0 to 4 A_0 in 4 seconds. The magnitude of this torque is :–

(1) $\frac{3A_0}{4}$ (2) A_0

(3) $4A_0$ (4) 12 A_0

50. The torque of force $\vec{F} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ newton acting at a point $\vec{r} = 3\hat{i} + 2\hat{j} + 3\hat{k}$ metre about origin is :-

(1) $6\hat{i} - 6\hat{j} + 12\hat{k}$ N – m

(2) $-6\hat{i} + 6\hat{j} - 12\hat{k}$ N – m

(3) $17\hat{i} - 6\hat{j} - 13\hat{k}$ N – m

(4) $-17\hat{i} + 6\hat{j} + 13\hat{k}$ N – m

51. When constant torque is acting on a body then :–

(1) body maintain its state or moves in straight line with same velocity

(2) acquire linear acceleration

(3) acquire angular acceleration

(4) rotates with a constant angular velocity

52. If torque on a body is zero, then which is conserved:

(1) force

(2) linear momentum

(3) angular momentum

(4) angular impulse

53. If I = 50 kg-m^2, then how much torque will be applied to stop it in 10 sec. Its initial angular speed is 20 rad/sec. :–

(1) 100 N-m (2) 150 N-m

(3) 200 N-m (4) 250 N-m

54. A force $\vec{F} = 2\hat{i} - 3\hat{k}$ acts on a particle at $\vec{r} = 0.5\hat{j} - 2\hat{k}$. The torque $\vec{\tau}$ acting on the particle relative to a point with co-ordinates (2.0 m, 0, –3.0 m) is

(1) $(-3.0\hat{i} - 4.5\hat{j} - \hat{k})$N – m

(2) $(3\hat{i} + 6\hat{j} - \hat{k})$N – m

(3) $(-20\hat{i} + 4.0\hat{j} + \hat{k})$N – m

(4) $(-1.5\hat{i} - 4.0\hat{j} - \hat{k})$N – m

55. If a ladder is not in balance against a smooth vertical wall, then it can be made in balance by :-

(1) Decreasing the length of ladder

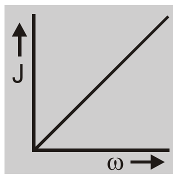
(2) Increasing the length of ladder

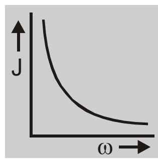
(3) Increasing the angle of inclination

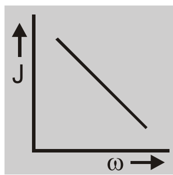
(4) Decreasing the angle of inclination

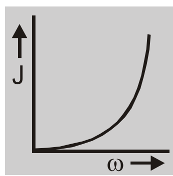
ANGULAR MOMENTUM

56. The graph between the angular momentum J and angular velocity ω for a body will be :-

(1) 

(2) 

(3) 

(4) 

57. When a mass is rotating in a plane about a fixed point, its angular momentum is directed along

(1) the radius

(2) the tangent to orbit

(3) line at an angle of 45° to the plane of rotation

(4) the axis of rotation

58. A ring of mass 10 kg and diameter 0.4 meter is rotating about its geometrical axis at 1200 rotations per minute. Its moment of inertia and angular momentum will be respectively:-

(1) 0.4 kg-m^2 and 50.28 J-s

(2) 0.4 kg-m^2 and 0.4 J-s

(3) 50.28 kg-m^2 and 0.4 J-s

(4) 0.4 kg-m^2 and zero

59. The rotational kinetic energy of two bodies of moments of ineritia 9 kg-m^2 and 1kg-m^2 are same. The ratio of their angular momentum is :-

(1) 3 : 1 (2) 1 : 3

(3) 9 : 1 (4) 1 : 9

60. A fan of moment of inertia 0.6 kgm^2 is to run upto a working speed of 0.5 revolution per second. Indicate the correct value of the angular momentum of the fan

(1) $0.6\pi\ \text{kg} \times \frac{\text{metre}^2}{\text{sec}}$ (2) $6\pi\ \text{kg} \times \frac{\text{metre}^2}{\text{sec}}$

(3) $3\pi\ \text{kg} \times \frac{\text{metre}^2}{\text{sec}}$ (4) $\frac{\pi}{6}\ \text{kg} \times \frac{\text{metre}^2}{\text{sec}}$

61. A body of mass m is moving with constant velocity parallel to x-axis. The angular momentum with respect to origin :–

(1) increases with time (2) decreases with time

(3) does not change (4) none of above

62. In an orbital motion, the angular momentum vector is :–

(1) along the radius vector

(2) parallel to the linear momentum

(3) in the orbital plane

(4) perpendicular to the orbital plane

63. Two bodies have their moments of inertia I and 2I respectively about their axis of rotation. If their kinetic energies of rotation are equal, their angular momentum will be in the ratio :–

(1) 1 : 2 (2) $\sqrt{2} : 1$ (3) $1 : \sqrt{2}$ (4) 2 : 1

64. The rotational kinetic enrgy of a body is K_{rot} and its moment of inertia is I. The angular momentum of body is

(1) IK_{rot} (2) $2\sqrt{IK_{rot}}$ (3) $\sqrt{2IK_{rot}}$ (4) $2IK_{rot}$

65. A body of mass 10 kg and radius of gyration 0.1 m is rotating about an axis. If angular speed is 10 rad/s, then angular momentum will be :–

(1) 1 kg m^2/s (2) 0.1 kg m^2/s

(3) 100 kg m^2/s (4) 10 kg m^2/s

66. A particle of mass 1.0 kg is rotating on a circular path of diameter 2.0 m at the rate of 10 rotations in 31.4s. The angular momentum of the body, (in kgm^2/s) is :-

(1) 1.0 (2) 1.5 (3) 2.0 (4) 4.0

67. A particle of mass m is rotating in a plane in a circular path of radius r. Its angular momentum is L. The centripetal force acting on the particle is

(1) $\frac{L^2}{mr}$ (2) $\frac{L^2m}{r}$ (3) $\frac{L^2}{mr^2}$ (4) $\frac{L^2}{mr^3}$

CONSERVATION OF ANGULAR MOMENTUM

68. A stone attached to one end of string is revolved around a stick so that the string winds upon the stick and gets shortened. What is conserved ?

(1) angular momentum

(2) linear momentum

(3) kinetic energy

(4) none of the above

69. A rotating table completes one rotation in 10 sec. and its moment of inertia is 100 kg-m^2. A person of 50 kg. mass stands at the centre of the rotating table. If the person moves 2m from the centre, the angular velocity of the rotating table (in rad/sec). will be:

(1) $\frac{2\pi}{30}$ (2) $\frac{20\pi}{30}$ (3) $\frac{2\pi}{3}$ (4) 2π

70. On melting of ice on the pole of the earth, its moment of inertia will :-

(1) increase (2) decrease

(3) remain unchanged (4) none of these

71. If the earth loses its atmosphere suddenly, then the duration of day will :-

(1) increase

(2) decrease

(3) remain unchanged

(4) nothing can be definitely said

72. A wheel is rotating about its axis at a constant angular velocity. If suddenly an object sticks to it on the rim, then its M.I. will :-
(1) increase
(2) decrease
(3) remain unchanged
(4) nothing can be said

73. In the above question, the angular velocity will :–
(1) increase
(2) decrease
(3) not change
(4) nothing can be said

74. An ant is sitting at the edge of a rotating disc. If the ant reaches the other end, after moving along the diameter, the angular velocity of the disc will:-
(1) remain constant
(2) first decreases and then increases
(3) first increases, then decrease
(4) Increase continuously

75. The angular momentum of body remains conserve if :–
(1) applied force on body is zero.
(2) applied torque on body is zero.
(3) applied force on body is constant.
(4) applied torque on body is constant.

76. A person is standing on the edge of a circular platform, which is moving with constant angular speed about an axis passing through its centre and perpendicular to the plane of platform. If person is moving along any radius towards axis of rotation then the angular velocity will :–
(1) decrease (2) remain unchanged
(3) increase (4) data is insufficient

77. A thin circular ring of mass M and radius 'r' is rotating about its axis with a constant angular velocity ω. Four objects each of mass m, are kept gently to the opposite ends of two perpendicular diameters of the ring. The angular velocity of the ring will be :–
(1) $\frac{M\omega}{4m}$ (2) $\frac{M\omega}{M+4m}$
(3) $\frac{(M+4m)\omega}{M}$ (4) $\frac{(M+4m)\omega}{M+4m}$

78. A round disc of moment of inertia I_2 about its axis perpendicular to its plane and passing through its centre is placed over another disc of moment of inertia I_1 rotating with an angular velocity ω about the same axis. The final angular velocity of the combination of discs is :–
(1) ω (2) $\frac{I_1\omega}{I_1+I_2}$
(3) $\frac{(I_1+I_2)\omega}{I_1}$ (4) $\frac{I_2\omega}{I_1+I_2}$

79. Rate of change of angular momentum with respect to time is proportional to :–
(1) angular velocity (2) angular acceleration
(3) moment of inertia (4) torque

80. A small steel sphere of mass m is tied to a string of length r and is whirled in a horizontal circle with a uniform angular velocity 2ω. The string is suddenly pulled, so that radius of the circle is halved. The new angular velocity will be
(1) 2ω (2) 4ω (3) 6ω (4) 8ω

81. If the earth were to suddenly contract to half its present size, without any change in its mass, the duration of the new day will be
(1) 18 hours (2) 30 hours
(3) 6 hours (4) 12 hours

82. A stone is attached to the end of a string and whirled in horizontal circle, then :–
(1) its linear and angular momentum are constant
(2) only linear momentum is constant
(3) its angular momentum is constant but linear momentum is variable
(4) both are variable

PURE ROLLING AND KINETIC ENERGY

83. A thin rod of length L is suspended from one end and rotated with n rotations per second. The rotational kinetic energy of the rod will be :
(1) $2mL^2\pi^2n^2$
(2) $\frac{1}{2}mL^2\pi^2n^2$
(3) $\frac{2}{3}mL^2\pi^2n^2$
(4) $\frac{1}{6}mL^2\pi^2n^2$

84. The rotational kinetic energy of a body is E. In the absence of external torque, if mass of the body is halved and radius of gyration doubled, then its rotational kinetic energy will be :-

(1) 0.5E (2) 0.25E (3) E (4) 2E

85. A particle of mass m is describing a circular path of radius r with uniform speed. If L is the angular momentum of the particle (about the axis of the circle), then the kinetic energy of the particle is

(1) $\frac{L^2}{mr^2}$ (2) mr^2L (3) $\frac{L^2}{2mr^2}$ (4) $\frac{L^2r^2}{m}$

86. A particle performs uniform circular motion with an angular momentum L. If the frequency of particle's motion is doubled and its kinetic energy halved, the angular momentum becomes

(1) 2L (2) 4L (3) $\frac{L}{2}$ (4) $\frac{L}{4}$

87. A flywheel is making $\frac{3000}{\pi}$ revolutions per minute about its axis. If the moment of inertia of the flywheel about that axis is 400 kgm^2, its rotational kinetic energy is

(1) 2×10^6 J (2) 3×10^3 J

(3) $500\pi^2$ J (4) 12×10^3 J

88. A ring is rolling without slipping. Its energy of translation is E. Its total kinetic energy will be :-

(1) E (2) 2E (3) 3E (4) 4E

89. A thin hollow cylinder open at both ends slides without rotating and then rolls without slipping with the same speed. The ratio of the kinetic energies in the two cases is

(1) 1 : 1 (2) 1 : 2

(3) 2 : 1 (4) 1 : 4

90. A solid sphere of mass M and radius R rolls on a horizontal surface without slipping. The ratio of rotational K.E. to total K.E. is :–

(1) $\frac{1}{2}$ (2) $\frac{3}{7}$

(3) $\frac{2}{7}$ (4) $\frac{2}{10}$

91. A disc is rolling on an inclined plane without slipping then what fraction of its total energy will be in form of rotational kinetic energy :–

(1) 1 : 3 (2) 1 : 2 (3) 2 : 7 (4) 2 : 5

92. A wheel is rolling along the ground with a speed of 2 m/s. The magnitude of the linear velocity of the points at the extermities of the horizontal diameter of the wheel is equal to

(1) $2\sqrt{10}\,m/s$ (2) $2\sqrt{3}\,m/s$

(3) $2\sqrt{2}\,m/s$ (4) 2 m/s

93. If rotational kinetic energy is 50% of total kinetic energy then the body will be :–

(1) ring (2) cylinder

(3) hollow sphere (4) solid sphere

94. A ball rolls without slipping. The radius of gyration of the ball about an axis passing through its centre of mass is K. If radius of the ball be R, then the fraction of total energy associated with its rotational energy will be :–

(1) $\frac{K^2+R^2}{R^2}$ (2) $\frac{K^2}{R^2}$

(3) $\frac{K^2}{K^2+R^2}$ (4) $\frac{R^2}{K^2+R^2}$

95. If the angular velocity of a body rotating about an axis is doubled and its moment of inertia halved, the rotational kinetic energy will changed by a factor of :-

(1) 4 (2) 2 (3) 1 (4) 1/2

96. If a sphere is rolling, the ratio of its rotational energy to the total kinetic energy is given by

(1) 7 : 10 (2) 2 : 5 (3) 10 : 5 (4) 2 : 7

97. A person, with outstretched arms, is spinning on a rotating stool. He suddenly brings his arms down to his sides. Which of the following is true about system kinetic energy K and angular momentum L ?

(1) Both K and L increase.

(2) Both K and L remain unchanged.

(3) K remains constant, L increases.

(4) K increases but L remains constant.

98. A uniform thin ring of mass 0.4 kg rolls without slipping on a horizontal surface with a linear velocity of 10 cm/s. The kinetic energy of the ring is :-

(1) 4×10^{-3} J

(2) 4×10^{-2} J

(3) 2×10^{-3} J

(4) 2×10^{-2} J

99. A body is rotating with angular momentum L. If I is its moment of inertia about the axis of rotation, its kinetic energy of rotation is :-

(1) $\frac{1}{2}IL^2$ (2) $\frac{1}{2}IL$

(3) $\frac{1}{2}(I^2/L)$ (4) $\frac{L^2}{2I}$

ROLLING MOTION ON AN INCLINED PLANE

100. A disc rolls down a plane of length L and inclined at angle θ, without slipping. Its velocity on reaching the bottom will be :-

(1) $\sqrt{\frac{4gL\sin\theta}{3}}$ (2) $\sqrt{\frac{2gL\sin\theta}{3}}$

(3) $\sqrt{\frac{10gL\sin\theta}{7}}$ (4) $\sqrt{4gL\sin\theta}$

101. A spherical shell and a solid cylinder of same radius rolls down an inclined plane. The ratio of their accelerations will be:-

(1) 15 : 14 (2) 9 : 10

(3) 2 : 3 (4) 3 : 5

102. A ring takes time t_1 and t_2 for sliding down and rolling down an inclined plane of length L respectively for reaching the bottom. The ratio of t_1 and t_2 is :-

(1) $\sqrt{2}$: 1 (2) 1 : $\sqrt{2}$

(3) 1 : 2 (4) 2 : 1

103. A solid sphere is rolling down on inclined plane from rest and a rectangular block of same mass is also slipping down simultaneously from rest on a similar smooth inclined plane.

(1) both of them will reach the bottom simultaneously

(2) the sphere will reach the bottom first

(3) the rectangular block will reach the bottom first.

(4) depends on density of material

104. Calculate the ratio of the times taken by a uniform solid sphere and a disc of the same mass and the same diameter to roll down through the same distance from rest on a inclined plane.

(1) 15 : 14 (2) $\sqrt{15} : \sqrt{14}$

(3) $15^2 : 14^2$ (4) $\sqrt{14} : \sqrt{15}$

105. A body of mass m slides down an incline and reaches the bottom with a velocity v. If the same mass were in the form of a ring which rolls down this incline, the velocity of the ring at the bottom would have been

(1) v (2) $v\sqrt{2}$

(3) $\frac{v}{\sqrt{2}}$ (4) $\left(\sqrt{\frac{2}{5}}\right)v$

106. When a sphere of moment of inertia I rolls down on an inclined plane the percentage of total energy which is rotational, is approximately

(1) 28 %

(2) 72 %

(3) 100 %

(4) none of these

107. When a body starts to roll on an inclined plane, its potential energy is converted into

(1) translational kinetic energy only

(2) translational and rotational kinetic energy

(3) rotational energy only

(4) none

108. A solid cylinder of mass M and radius R rolls without slipping down an inclined plane of length L and height h. What is the speed of its centre of mass when the cylinder reaches its bottom :–

(1) $\sqrt{2gh}$ (2) $\sqrt{\frac{3}{4}gh}$

(3) $\sqrt{\frac{4}{3}gh}$ (4) $\sqrt{4gh}$

109. Which of the following is true about the angluar momentum of a cylinder rolling down a slope without slipping ?

(1) Its magnitude changes but the direction remains same

(2) both magnitude and direction change

(3) only the direction change

(4) neither change

110. A sphere and a disc of same radii and mass are rolling on an inclined plane without slipping. a_s & a_d are acceleration and g is acceleration due to gravity. Then which statement is correct ?

(1) $a_s > a_d > g$ (2) $g > a_s > a_d$

(3) $a_s > g > a_d$ (4) $a_d > a_s > g$

EXERCISE-I (Conceptual Questions) **ANSWER KEY**

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	4	2	1	4	4	1	2	4	2	2	1	3	3	3	4
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Ans.	2	3	4	4	1	2	2	1	3	1	2	2	2	2	2
Que.	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
Ans.	3	4	2	4	4	2	3	2	2	1	2	3	2	2	2
Que.	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
Ans.	4	1	2	1	3	3	3	1	4	3	1	4	1	1	1
Que.	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
Ans.	3	4	3	3	1	3	4	1	1	1	2	1	2	3	2
Que.	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
Ans.	3	2	2	4	4	3	3	3	1	3	4	1	2	2	3
Que.	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105
Ans.	1	3	1	3	2	4	4	1	4	1	2	2	3	4	3
Que.	106	107	108	109	110										
Ans.	1	2	3	1	2										

EXERCISE-II (Assertion & Reason)

Directions for Assertion & Reason questions

These questions consist of two statements each, printed as Assertion and Reason. While answering these Questions you are required to choose any one of the following four responses.

(A) If both Assertion & Reason are True & the Reason is a correct explanation of the Assertion.

(B) If both Assertion & Reason are True but Reason is not a correct explanation of the Assertion.

(C) If Assertion is True but the Reason is False.

(D) If both Assertion & Reason are false.

1. ***Assertion:*** If earth were to shrink, length of the day would increase.

Reason : Smaller objects would take more time to complete one rotation around its axis.

(1) A (2) B (3) C (4) D

2. ***Assertion:*** Only rotating bodies can have angular momentum.

Reason : The perpendicular axis theorem only applicable for the axis passing through the centre of mass of the body.

(1) A (2) B (3) C (4) D

3. ***Assertion*** : A couple does not exert a net force on an object even though it exerts a torque.

Reason : Couple is a pair of two forces with equal magnitude but opposite directions acting simultaneously on a body in different lines of action.

(1) A (2) B (3) C (4) D

4. ***Assertion:*** The total distance moved by any point on the periphery of a wheel of radius R along the surface in one revolution is $2\pi R$.

Reason : In rolling motion of a wheel, every point on its periphery comes in contact with the surface once in one revolution.

(1) A (2) B (3) C (4) D

5. ***Assertion:*** To unscrew a rusted nut, we need a pipe wrench with longer arm.

Reason : Wrench with longer arm reduces the force applied on the arm.

(1) A (2) B (3) C (4) D

6. ***Assertion:*** The condition of equilibrium for a rigid body is -

Translational equilibrium : $\sum \vec{F} = 0$, (i.e. sum of all external forces equal to zero.)

Rotational equilibrium : $\sum \vec{\tau} = 0$, (i.e. sum of all external torques equal to zero.)

Reason : A rigid body must be in equilibrium under the action of two equal and opposite forces.

(1) A (2) B (3) C (4) D

7. ***Assertion :*** For the purpose of calculation of moment of inertia, body's mass can be assumed to be concentrated at its centre of mass.

Reason : Moment of inertia of rigid body about an axis passing through its centre of mass is zero.

(1) A (2) B (3) C (4) D

8. ***Assertion :*** Many great rivers flows toward the equator. The small particle that they carry increases the time of rotation of the earth about its own axis.

Reason : The angular momentum of the earth about its rotation axis is conserved.

(1) A (2) B (3) C (4) D

9. ***Assertion :*** The spokes near the top of a rolling bicycle wheel are more blurred than those near the bottom of the wheel.

Reason : The spokes near the top of wheel are moving faster than those near the bottom of the wheel.

(1) A (2) B (3) C (4) D

10. ***Assertion :*** A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling).

Reason : For pure rolling, work done against frictional force is zero.

(1) A (2) B (3) C (4) D

11. ***Assertion :-*** Angular momentum may not necessarily be parallel to angular velocity vector.

Reason :- The body may not be symmetrical about its axis of rotation.

(1) A (2) B (3) C (4) D

12. ***Assertion :-*** As star collapse its angular velocity increases.

Reason :- The mass of star decreases

(1) A (2) B (3) C (4) D

13. ***Assertion :-*** Rigid body can be elastic.

Reason :- If a force is applied on the rigid body, its dimension may change. **[AIIMS 2017]**

(1) A (2) B (3) C (4) D

14. ***Assertion :-*** A sharp needle when balanced on a edge falls even under a small disturbance.

Reason :- Small disturbance causes torque in the same direction as that of angular displacement.

[AIIMS 2017]

(1) A (2) B (3) C (4) D

15. ***Assertion :-*** A rotating body can be in stable and unstable equilibrium.

Reason :- Moment of inertia is always different for different axis of rotation. **[AIIMS 2017]**

(1) A (2) B (3) C (4) D

16. ***Assertion :-*** Practically a body rolling on a surface will loose its complete K.E. **[AIIMS 2017]**

Reason :- Friction will convert its K.E. into P.E. of atoms.

(1) A (2) B (3) C (4) D

17. ***Assertion :-*** Spin angular momentum is intrinsic property of body. **[AIIMS 2018]**

Reason :- It doesn't depend on coordinates of body.

(1) A (2) B (3) C (4) D

18. ***Assertion :-*** Torque on a body can be zero even if there is a net force on it. **[AIIMS 2018]**

Reason :- Torque and force on a body are always perpendicular.

(1) A (2) B (3) C (4) D

19. ***Assertion :-*** Torque on a body may be non-zero even if net force acting on it is zero. **[AIIMS 2018]**

Reason :- Force does not depend on point of application, but torque depends on point of application.

(1) A (2) B (3) C (4) D

EXERCISE-II (Assertion & Reason) **ANSWER KEY**

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	4	4	1	1	1	3	4	1	1	2	1	3	4	1	3
Que.	16	17	18	19											
Ans.	1	1	2	1											